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# RELATIONS AND FUNCTIONS

## INTRODUCTION

In daily life, we come across many relations such as Teacher and Student, Mother and Daughter, Book and Cost. In mathematics also, we come across many relations such as

(*i*) number $x$ is square of number $y$

(*ii*) line $l$ is perpendicular to line $m$

(*iii*) set A is a proper subset of set B

(*iv*) area of a circle with radius $r$ is $\pi r^2$.

In each of these, we notice that a relation involves pairs of objects in a certain order. In this chapter, we will learn how to connect pairs of objects from two sets and then introduce relation between two objects of the pair. Finally, we shall learn about special type of relations called functions. From the beginning of modern mathematics in the 17th century, the concept of function has been at the very centre of mathematical thought. It gives the mathematical rule by which one quantity corresponds to another quantity.

## 2.1 ORDERED PAIR

*An **ordered pair** is a pair of objects taken in a specific order.*

An ordered pair is written by listing its two members in a specific order, separating them by a comma and enclosing the pair in parentheses. In the ordered pair $(a, b)$, $a$ is called the *first member* (or *component*) and $b$ is called the *second member* (or *component*).

**Equality of ordered pairs.** *Two ordered pairs $(a, b)$ and $(c, d)$ are called **equal**, written as $(a, b) = (c, d)$, iff $a = c$ and $b = d$.*

**REMARKS**

1. The word ordered implies that the order in which the two elements of the pair occur is meaningful. For example, if we have a sock and a shoe, the order in which they are put on matters. In fact, there are situations in which order is very important and essential.
2. The ordered pairs $(a, b)$ and $(b, a)$ are different unless $a = b$.
3. The two components of an ordered pair may be equal.
4. Note that $\{a, b\} \neq (a, b)$, because $\{a, b\}$ is a set whereas $(a, b)$ is an ordered pair.

## 2.2 CARTESIAN PRODUCT OF TWO SETS

*Let A and B be any two non-empty sets, then the set of all ordered pairs $(a, b)$ for all $a \in A$ and $b \in B$ is called the **cartesian product** of A and B. It is written as $A \times B$ (read as 'A cross B').*

*Symbolically, $A \times B = \{(a, b) : \text{for all } a \in A, b \in B\}$.*

For example, let A = {1, 2, 3} and B = {3, 4}, then
A × B = {(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)} and
B × A = {(3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3)}.

From this example, we observe that

(*i*) A × B ≠ B × A. (*ii*) $n$(A × B) = 6 = $n$(B × A).

(*iii*) $n$(A × B) = 6 = 3 × 2 = $n$(A) × $n$(B).

**REMARK**

1. A × B ≠ B × A unless A = B.
2. If A and B are finite sets, then
$$n(A \times B) = n(A) \times n(B) \text{ and } n(A \times B) = n(B \times A).$$
3. A × B = φ when one or both of A, B are empty sets.
4. A × B ≠ φ iff A ≠ φ and B ≠ φ.
5. If A and B are non-empty sets and either A or B is an infinite set, then A × B is also infinite set.

The concept of cartesian product can be extended to more than two sets.

*Let A, B and C be any non-empty sets, then the set of all triplets (a, b, c) for all a ∈ A, b ∈ B and c ∈ C is called the* ***cartesian product*** *of A, B and C. It is written as A × B × C. Thus,*
$$A \times B \times C = \{(a, b, c) : \text{for all } a \in A, b \in B, c \in C\}.$$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *If the ordered pairs* $(x - 1, y + 3)$ *and* $(2, x + 4)$ *are equal, find* $x$ *and* $y$.

**Solution.** $(x - 1, y + 3) = (2, x + 4)$

⇒ $x - 1 = 2$ and $y + 3 = x + 4$

⇒ $x = 3$ and $y = x + 1$

⇒ $x = 3$ and $y = 3 + 1 = 4$.

Hence, $x = 3$ and $y = 4$.

**Example 2.** *Find* $x$ *and* $y$ *if* $(x^2 - 3x, y^2 + 4y) = (-2, 5)$.

**Solution.** Given $(x^2 - 3x, y^2 + 4y) = (-2, 5)$

⇒ $x^2 - 3x = -2$ and $y^2 + 4y = 5$

⇒ $x^2 - 3x + 2 = 0$ and $y^2 + 4y - 5 = 0$

⇒ $(x - 1)(x - 2) = 0$ and $(y - 1)(y + 5) = 0$

⇒ $x = 1, 2$ and $y = 1, -5$.

Hence, $x = 1, 2$; $y = 1, -5$.

**Example 3.** *If P = {a, b, c} and Q = {d}, form the sets P × Q and Q × P. Are these two cartesian products equal?*

**Solution.** Given P = {$a$, $b$, $c$} and Q = {$d$}, by definition of certesian product, we get

P × Q = {($a$, $d$), ($b$, $d$), ($c$, $d$)}

and Q × P = {($d$, $a$), ($d$, $b$), ($d$, $c$)}.

By definition of equality of ordered pairs, the pair ($a$, $d$) is not equal to the pair ($d$, $a$), therefore, P × Q ≠ Q × P.

**Example 4.** *If A = {p, q} and B = {x : x ∈ N and x is a prime number less than 6}, then find A × B.*

**Solution.** Given A = {$p$, $q$} and

B = {$x$ : $x$ ∈ **N** and $x$ is a prime number less than 6}
= {2, 3, 5}.

Then, A × B = {($p$, 2), ($p$, 3), ($p$, 5), ($q$, 2), ($q$, 3), ($q$, 5)}.

**Example 5.** *If $A = \{1, 2, 3, 4\}$ and $x, y \in A$, form the set of all ordered pairs $(x, y)$ such that $x$ is a divisor of $y$.*

**Solution.** Given $A = \{1, 2, 3, 4\}$ and $x, y \in A$.

The set of all ordered pairs $(x, y)$ such that $x$ is a divisor of $y$

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

**Example 6.** *Express $\{(x, y) : y + 2x = 5, x, y \in W\}$ as the set of ordered pairs.*

**Solution.** Since $y + 2x = 5$ and $x, y \in W$,

put $x = 0$, $y + 0 = 5 \Rightarrow y = 5$,

$x = 1$, $y + 2 \times 1 = 5 \Rightarrow y = 3$,

$x = 2$, $y + 2 \times 2 = 5 \Rightarrow y = 1$.

For all other values of $x \in W$, we do not get $y \in W$.

Hence, the required set of ordered pairs is $\{(0, 5), (1, 3), (2, 1)\}$.

**Example 7.** *Express $\{(x, y) : x + 3y = 20, x, y \in N\}$.*

**Solution.** Given $x + 3y = 20 \Rightarrow x = 20 - 3y$, $x, y \in N$.

When $y = 1$, $x = 20 - 3 \times 1 = 17$;

when $y = 2$, $x = 20 - 3 \times 2 = 14$;

when $y = 3$, $x = 20 - 3 \times 3 = 11$;

when $y = 4$, $x = 20 - 3 \times 4 = 8$;

when $y = 5$, $x = 20 - 3 \times 5 = 5$;

when $y = 6$, $x = 20 - 3 \times 6 = 2$.

For all other values of $y \in N$, we do not get $x \in N$.

Hence, the required set of ordered pairs is

$\{(2, 6), (5, 5), (8, 4), (11, 3), (14, 2), (17, 1)\}$.

**Example 8.** *If $A = \{1, 5\}$, $B = \{2, 6\}$, $C = \{2, 4\}$, find $A \times (B \cup C)$.*

**Solution.** Given $A = \{1, 5\}$, $B = \{2, 6\}$, $C = \{2, 4\}$,

then $B \cup C = \{2, 4, 6\}$.

$\therefore$ $A \times (B \cup C) = \{(1, 2), (1, 4), (1, 6), (5, 2), (5, 4), (5, 6)\}$.

**Example 9.** *If $P = \{x : x < 3, x \in N\}$ and $Q = \{x : x \le 2, x \in W\}$, find $(P \cup Q) \times (P \cap Q)$.*

**Solution.** Given $P = \{x : x < 3, x \in N\} = \{1, 2\}$ and

$Q = \{x : x \le 2, x \in W\} = \{0, 1, 2\}$

$\Rightarrow$ $P \cup Q = \{0, 1, 2\}$ and $P \cap Q = \{1, 2\}$.

$\therefore$ $(P \cup Q) \times (P \cap Q) = \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2)\}$.

**Example 10.** *If $A = \{x \mid x \in W, x < 3\}$, $B = \{x \mid x \in N, 2 \le x < 4\}$ and $C = \{3, 4\}$, then verify that*

$(A \cup B) \times C = (A \times C) \cup (B \times C)$.

**Solution.** Given $A = \{x \mid x \in W, x < 3\} = \{0, 1, 2\}$

$B = \{x \mid x \in N, 2 \le x < 4\} = \{2, 3\}$ and $C = \{3, 4\}$

$\Rightarrow$ $A \cup B = \{0, 1, 2, 3\}$.

$\therefore$ $(A \cup B) \times C = \{(0, 3), (0, 4), (1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$ ...(*i*)

$A \times C = \{(0, 3), (0, 4), (1, 3), (1, 4), (2, 3), (2, 4)\}$ and

$B \times C = \{(2, 3), (2, 4), (3, 3), (3, 4)\}$.

$\therefore$ $(A \times C) \cup (B \times C) = \{(0, 3), (0, 4), (1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$ ...(*ii*)

From (*i*) and (*ii*), we find that

$(A \cup B) \times C = (A \times C) \cup (B \times C)$.

**Example 11.** *If* $A = \{1, 2\}$, $B = \{2, 3\}$ *and* $C = \{0\}$, *form the set* $A \times B \times C$.

**Solution.** Given $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{0\}$, by def.,

$$A \times B \times C = \{(1, 2, 0), (1, 3, 0), (2, 2, 0), (2, 3, 0)\}.$$

**Example 12.** *If* $A = \{-1, 1\}$, *form the set* $A \times A \times A$.

**Solution.** Given $A = \{-1, 1\}$, by def.,

$$A \times A \times A = \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}.$$

**Example 13.** *If* **R** *is the set of all real numbers, what do the cartesian products* **R** × **R** *and* **R** × **R** × **R** *represent?*

**Solution.** The cartesian product **R** × **R** represents the set $\{(x, y) : x, y \in \mathbf{R}\}$ which represents the co-ordinates of all points in two dimensional space.

The cartesian product **R** × **R** × **R** represents the set $\{(x, y, z) : x, y, z \in \mathbf{R}\}$ which represents the co-ordinates of all points in three dimensional space.

**Example 14.** *If* $A \times B = \{(0, 2), (3, -1), (4, 2), (0, -1), (3, 2), (4, -1)\}$, *then find* $B \times A$.

**Solution.** Clearly, $B \times A$ can be obtained from $A \times B$ by interchanging the components of ordered pairs in $A \times B$.

$\therefore$ $B \times A = \{(2, 0), (-1, 3), (2, 4), (-1, 0), (2, 3), (-1, 4)\}$.

**Example 15.** *If* $A \times B = \{(a, p), (b, q), (c, p), (a, q), (b, p), (c, q)\}$, *find A and B.*

**Solution.** A = set of first components of $A \times B = \{a, b, c\}$,

B = set of second components of $A \times B = \{p, q\}$.

**Example 16.** *Let A and B be two sets such that* $n(A) = 3$ *and* $n(B) = 2$. *If* $(x, 1), (y, 2), (z, 1)$ *are in* $A \times B$, *find A and B, where x, y, z are distinct elements.*

**Solution.** Since $x$, $y$ and $z$ are distinct elements and $(x, 1), (y, 2), (z, 1)$ are elements of $A \times B$, therefore,

$x, y, z \in A$ and $1, 2 \in B$.

But $n(A) = 3$ and $n(B) = 2$,

$\therefore$ $A = \{x, y, z\}$ and $B = \{1, 2\}$.

**Example 17.** *The cartesian product* $A \times A$ *has 9 elements among which are found* $(-1, 0)$ *and* $(0, 1)$. *Find the set A and the remaining elements of* $A \times A$.

**Solution.** Let $n(A) = m$.

Given $n(A \times A) = 9 \Rightarrow n(A) \cdot n(A) = 9$

$\Rightarrow$ $m \cdot m = 9 \Rightarrow m^2 = 9 \Rightarrow m = 3$ $\quad (\because m > 0)$

Given $(-1, 0) \in A \times A \Rightarrow -1 \in A$ and $0 \in A$.

Also $(0, 1) \in A \times A \Rightarrow 0 \in A$ and $1 \in A$.

Thus, $-1, 0, 1 \in A$ but $n(A) = 3$.

Therefore, $A = \{-1, 0, 1\}$.

The remaining elements of $A \times A$ are $(-1, -1), (-1, 1), (0, -1), (0, 0), (1, -1), (1, 0), (1, 1)$.

**Example 18.** *Given* $A = \{1, 2, 3, 4, 5\}$ *and* $R = \{(x, y) : x \in A, y \in A\}$. *Find the set of ordered pairs which satisfy the conditions given below :*

(*i*) $x + y = 5$ (*ii*) $x + y < 5$ (*iii*) $x + y > 8$.

**Solution.** Given $A = \{1, 2, 3, 4, 5\}$ and $R = \{(x, y) : x \in A, y \in A\}$

*i.e.* $R = A \times A$. First, we find R *i.e.* $A \times A$.

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5)\}$.

Thus, R has 25 different ordered pairs.

(*i*) The ordered pairs of R which satisfy the condition $x + y = 5$ are
(1, 4), (2, 3), (3, 2), (4, 1).

(*ii*) The ordered pairs of R which satisfy the condition $x + y < 5$ are
(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1).

(*iii*) The ordered pairs of R which satisfy the condition $x + y > 8$ are
(4, 5), (5, 4), (5, 5).

## EXERCISE 2.1

*Very short answer type questions (1 to 17) :*

1. Find $a$ and $b$ if
(*i*) $(a + 1, b - 2) = (3, 1)$ (*ii*) $\left(\frac{a}{3}+1, b-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$
(*iii*) $(2a, a + b) = (6, 2)$ (*iv*) $(a + b, 3b - 2) = (7, -5)$.
2. Find $x$ and $y$ if
(*i*) $(x - y, x + y) = (6, 10)$ (*ii*) $(2x + y, x - y) = (8, 3)$.
3. If the ordered pairs $(a, -1)$ and $(5, b)$ belong to $\{(x, y) : y = 2x - 3\}$, find the values of $a$ and $b$.
4. If P = {7, 8} and Q = {5, 4, 2}, find P × Q and Q × P.
5. If A = {–1, 0, 1} and B = {3, 5}, write the following :
(*i*) A × B (*ii*) B × A (*iii*) B × B.
6. If $n(A) = 2$ and B = {–1, 0, 3}, then what is number of elements in A × B?
7. If A is a set such that $n(A) = 3$ and B = {3, 4, 5}, then what is the number of elements in A × B ?
8. If A = {–3, –1, 0, 4} and B = {–1, 0, 1, 2, 3}, then write the number of elements in each of the following cartesian products :
(*i*) A × B (*ii*) B × A (*iii*) A × A (*iv*) B × B.
9. If A = {1, 2} and B = {3, 4}, then how many subsets will A × B have?
10. If $x \in \{2, 3, 5\}$ and $y \in \{2, 4, 6\}$, form the set of all ordered pairs $(x, y)$ such that $x < y$.
11. If $x \in \{-1, 2, 3, 4, 5\}$ and $y \in \{0, 3, 6\}$, form the set of all ordered pairs $(x, y)$ such that $x + y = 5$.
12. If $x \in \{2, 3, 4\}$ and $y \in \{4, 6, 9, 10\}$, form the set of all ordered pairs $(x, y)$ such that $x$ is a factor of $y$.
13. If A = {–1, 0, 1, 2, 3}, write the subset S of A × A such that the second component of the elements of S is 0.
14. If A × B = {($p$, $q$), ($p$, $r$), ($m$, $q$), ($m$, $r$)}, find A and B.
15. If A × B = {(–1, 1), (–1, 2), (2, 1), (2, 2), (3, 1), (3, 2)}, find A and B.
16. If A = {$x$, $y$, $z$} and some elements of A × B are ($x$, 1), ($y$, 2), ($z$, 1), then write the set B such that $n(A \times B) = 6$.
17. If A × B = {($x$, 1), ($y$, 2), ($x$, 3), ($y$, 3), ($y$, 1), ($x$, 2)}, then find B × A.
18. If A = {1, 2, 3, 4} and B = {5, 7, 9}, find
(*i*) A × B (*ii*) B × A
(*iii*) Is A × B = B × A? (*iv*) Is $n(A \times B) = n(B \times A)$?

19. Let A = {1, 2, 3}, B = {3, 4} and C = {4, 5, 6}. Find
(i) A × (B ∩ C) (ii) (A × B) ∩ (A × C)
(iii) A × (B ∪ C) (iv) (A × B) ∪ (A × C).

20. Let A = {1, 2}, B = {1, 2, 3, 4}, C = {5, 6} and D = {5, 6, 7, 8}.
Verify that
(i) A × (B ∩ C) = (A × B) ∩ (A × C) (ii) A × C is a subset of B × D.

21. If A = {$x$ : $x \in$ W, $x$ < 2}, B = {$x$ : $x \in$ N, 1 < $x$ < 5} and C = {3, 5}, find
(i) A × (B ∩ C) (ii) A × (B ∪ C).

22. If A = {$x$ : $x \in$ N and $x \le 3$}, B = {$x$ : $x \in$ I, $-1 \le x \le 1$} and C = {1, 2}, verify that
(i) A × (B ∪ C) = (A × B) ∪ (A × C) (ii) A × (B ∩ C) = (A × B) ∩ (A × C)
(iii) (A − B) × C = A × C − B × C.

23. If P = {1, 2}, form the set P × P × P.

24. If A, B are two sets such that $n$ (A × B) = 6 and some elements of A × B are (−1, 2), (2, 3), (4, 3), then find A × B and B × A.

25. Given B = {2, 3, 5} and some elements of A × B are ($a$, 2), ($b$, 3), ($c$, 5). Find the set A and the remaining ordered pairs of A × B such that A × B is least.

## 2.3 RELATIONS

In everyday life, we frequently speak of relations between two or more objects. To learn the concept properly, consider the following examples :

(i) Let A = {1, 2, 3, 5} and B = {2, 4}, then

A × B = {(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4), (5, 2), (5, 4)}.

We can obtain a subset of A × B by introducing a relation 'is less than' between the elements of the sets A and B.

If we write R for the relation '*is less than*', then we get

1 R 2, 1 R 4, 2 R 4, 3 R 4.

Omitting the letter R between the above pairs of numbers and writing these pairs of numbers as ordered pairs, the above information can be written as a set of ordered pairs R where

R = {(1, 2), (1, 4), (2, 4), (3, 4)}
= {($x$, $y$) : $x \in$ A, $y \in$ B, $x < y$}.

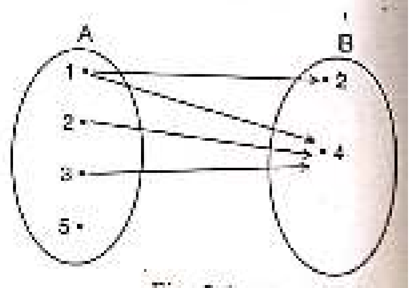


*Fig. 2.1.*

Thus, the relation '*is less than*' from the set A to the set B gives rise to a subset R of A × B such that ($x$, $y$) ∈ R iff $x$ R $y$ *i.e.* iff $x < y$.

(ii) Let A = {2, 3, 5, 9} and B = {4, 6, 9, 15, 25}. There is a relation '*is a divisor of*' between the elements of the sets A and B. If we write R for the relation '*is a divisor of*', then we get

2 R 4, 2 R 6, 3 R 6, 3 R 9, 3 R 15, 5 R 15, 5 R 25, 9 R 9.

This can be written as a set of ordered pairs R where

R = {(2, 4), (2, 6), (3, 6), (3, 9), (3, 15), (5, 15), (5, 25), (9, 9)}
= {($x$, $y$) : $x \in$ A, $y \in$ B, $x$ is a divisor of $y$}.

Thus, the relation '*is a divisor of*' from the set A to the set B gives rise to a subset R of A × B such that ($x$, $y$) ∈ R iff $x$ R $y$ *i.e.* iff $x$ is a divisor of $y$.

(iii) Let N be the set of natural numbers. There is a relation '*has as its square*' from the set N to N. If we write R for the relation '*has as its square*', then we get

1 R 1, 2 R 4, 3 R 9, 4 R 16, 5 R 25, ......

This can be written as a set of ordered pairs R where

$R = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25), \ldots\ldots\}$

$= \{(x, y) : x, y \in \mathbf{N}, y = x^2\}$.

Thus, R is a subset of N × N such that $(x, y) \in R$ iff $x R y$ *i.e.* iff $y = x^2$.

The above examples lead to :

**Definition.** *If A, B are any two (non-empty) sets, then any subset of A × B is called a* ***relation from*** *A to B.*

Let R be a relation from A to B. If R = ϕ, then R is called the *empty relation* and if R = A × B, then R is called the *universal relation.*

If R is a relation from A to B and if $(a, b) \in R$, then we write $a R b$ and say that $a$ is related to $b$ and if $(a, b) \notin R$, then we write $a \not R b$ and say that $a$ is not related to $b$.

*In particular, if A is any (non-empty) set, then any subset of A × A is called a* ***relation on*** *A.*

## 2.3.1 Representation of a relation

**1. Roster form.** In this form, a relation is represented by the set of all ordered pairs which belong to the given relation.

For example, let A = {1, 2, 3, 4, 5} and B = {1, 2, 3, 4, ..., 20}, and let R be the relation 'has as its square' from A to B, then

$R = \{(1, 1), (2, 4), (3, 9), (4, 16)\}$.

**2. Set-builder form.** In this form, the relation is represented as $\{(x, y) : x \in A, y \in B, x \ldots y\}$, the blank is to be replaced by the rule which associates $x$ and $y$.

For example, let A = {1, 3, 4, 5, 7}, B = {2, 4, 6, 8} and

R = {(1, 2), (3, 4), (5, 6), (7, 8)} then R in the builder form can be written as

$R = \{(x, y) : x \in A, y \in B, x$ is one less than $y\}$.

**3. By arrow diagram.** In this form, the relation is represented by drawing arrows from first components to the second components of all ordered pairs which belong to the given relation.

For example, let A = {1, 2, 3, 5}, B = {2, 3, 4} and R be the relation 'is greater than' from A to B, then

$R = \{(3, 2), (5, 2), (5, 3), (5, 4)\}$.

This relation R from A to B can be represented by the arrow diagram shown in fig. 2.2.

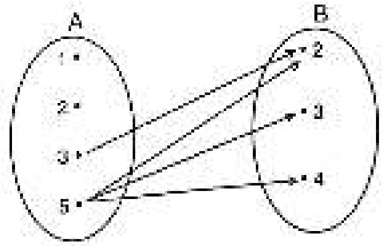



Fig. 2.2.

## 2.3.2 Domain and range of a relation

*Let A, B be any two (non-empty) sets and R be a relation from A to B, then the domain of the relation R, is the set of all first components of the ordered pairs which belong to R, and the range of the relation R is the set of all second components of the ordered pairs which belong to R. Thus,*

***domain of*** $R = \{x : x \in A, (x, y) \in R$ *for some* $y \in B\}$ *and*

***range of*** $R = \{y : y \in B, (x, y) \in R$ *for some* $x \in A\}$.

*If R is a relation from A to B, then B is called* ***codomain*** *of R.*

For example,

let A = {1, 3, 4, 5, 7}, B = {2, 4, 6, 8} and R be the relation *'is one less than'* from A to B, then

R = {(1, 2), (3, 4), (5, 6), (7, 8)}. Here,

domain of R = {1, 3, 5, 7} and range of R = {2, 4, 6, 8}.

In this example, note that range of R = B = codomain of R.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *If A and B are finite sets such that $n(A) = m$ and $n(B) = k$, find the number of relations from A to B.*

**Solution.** Given $n(A) = m$ and $n(B) = k$

$\therefore$ $n(A \times B) = n(A) \times n(B) = mk$.

$\therefore$ The number of subsets of $A \times B = 2^{mk}$

($\because$ if $n(A) = m$, then the number of subsets of $A = 2^m$)

Since every subset of $A \times B$ is a relation from A to B, therefore, the number of relations from A to B $= 2^{mk}$.

**Example 2.** *If a relation $R = \{(0, 0), (2, 4), (-1, -2), (3, 6), (1, 2)\}$, then*

(*i*) *write domain of R.* (*ii*) *write range of R.*

(*iii*) *write R in the builder form.* (*iv*) *represent R by an arrow diagram.*

**Solution.** Given $R = \{(0, 0), (2, 4), (-1, -2), (3, 6), (1, 2)\}$.

(*i*) Domain of $R = \{0, 2, -1, 3, 1\}$.

(*ii*) Range of $R = \{0, 4, -2, 6, 2\}$.

(*iii*) R in the builder form can be written as

$R = \{(x, y) : x \in I, -1 \leq x \leq 3, y = 2x\}$.

(*iv*) The relation R can be represented by the arrow diagram shown in fig. 2.3.

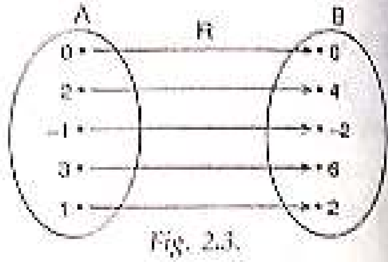



*Fig. 2.3.*

**Example 3.** *If $A = \{-1, 2, 5, 8\}$, $B = \{0, 1, 3, 6, 7\}$ and R be the relation 'is one less than' from A to B, then*

(*i*) *find R as a set of ordered pairs.* (*ii*) *find domain and range of R.*

**Solution.** (*i*) Given $A = \{-1, 2, 5, 8\}$, $B = \{0, 1, 3, 6, 7\}$ and R is the relation '*is one less than*' from A to B, therefore,

$R = \{(-1, 0), (2, 3), (5, 6)\}$.

(*ii*) Domain of $R = \{-1, 2, 5\}$ and range of $R = \{0, 3, 6\}$.

**Example 4.** *If $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4\}$ and $R = \{(x, y) : (x, y) \in A \times B, y = x + 1\}$, then*

(*i*) *find $A \times B$.* (*ii*) *write R in roster form.*

(*iii*) *write domain and range of R.* (*iv*) *represent R by an arrow diagram.*

**Solution.** (*i*) $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4)\}$.

(*ii*) $R = \{(1, 2), (2, 3), (3, 4)\}$.

(*iii*) Domain of $R = \{1, 2, 3\}$ and range of $R = \{2, 3, 4\}$.

(*iv*) The relation R can be represented by the arrow diagram shown in fig. 2.4.

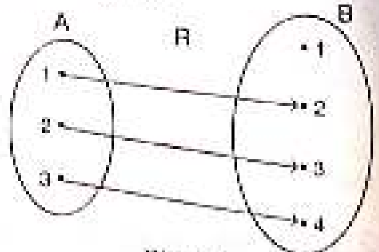



*Fig. 2.4.*

**Example 5.** *If $A = \{1, 2, 3, 4, \ldots, 14\}$ and a relation R is defined from A to A by*

$R = \{(x, y) : 3x - y = 0, x, y \in A\}$.

(*i*) *Write R in roster form.* (*ii*) *Write its domain, codomain and range.*

(*iii*) *Depict this relationship by an arrow diagram.*

**Solution.** Given $A = \{1, 2, 3, 4, \ldots, 14\}$, relation R from A to A is defined by $3x - y = 0$ *i.e.* $y = 3x$, $x, y \in A$.

(*i*) $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$.

(*ii*) Domain $= \{1, 2, 3, 4\}$,

codomain $= \{1, 2, 3, \ldots, 14\} = A$

and range $= \{3, 6, 9, 12\}$.

(*iii*) The relation R can be represented by the arrow diagram shown in fig. 2.5.

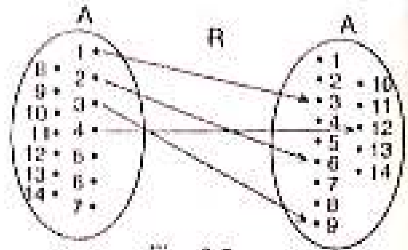



*Fig. 2.5.*

**Example 6.** *Define a relation R on the set N of natural numbers by*

$R = \{(x, y) : y = x + 5, x < 4, x, y \in N\}$. *Depict this relationship using*

*(i) roster form* *(ii) an arrow diagram*

*Write down the domain and range of R.*

**Solution.** (*i*) Given $R = \{(x, y) : y = x + 5, x < 4, x, y \in N\}$

As $x < 4$ and $x \in N$, so $x = 1, 2, 3$.

When $x = 1$, $y = 1 + 5 = 6$;

when $x = 2$, $y = 2 + 5 = 7$ and

when $x = 3$, $y = 3 + 5 = 8$.

$\therefore$ $R = \{(1, 6), (2, 7), (3, 8)\}$.

(*ii*)

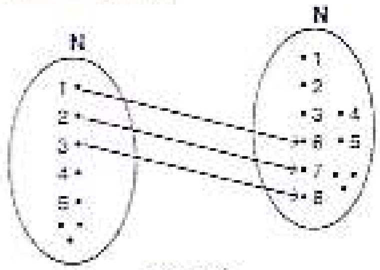



*Fig. 2.6.*

Domain = {1, 2, 3} and range = {6, 7, 8}

**Example 7.** *The adjoining diagram shows a relation between the sets P and Q. Write this relation*

*(i) in roster form*

*(ii) in set builder form.*

*What is its domain and range?*

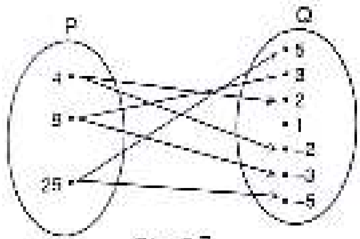



*Fig. 2.7.*

**Solution.** (*i*) In roster form,

$R = \{(4, 2), (4, -2), (9, 3), (9, -3), (25, 5), (25, -5)\}$

(*ii*) In set builder form, $R = \{(x, y) : x = y^2, x \in P, y \in Q\}$

Domain of this relation = {4, 9, 25} and its range = {2, –2, 3, –3, 5, –5}.

**Example 8.** *Find the domain and the range of the relation R defined by*

$R = \{(x + 1, x + 3) : x \in \{0, 1, 2, 3, 4, 5\}\}$.

**Solution.** Given $x \in \{0, 1, 2, 3, 4, 5\}$,

when $x = 0$, $x + 1 = 0 + 1 = 1$ and $x + 3 = 0 + 3 = 3$,

$x = 1$, $x + 1 = 1 + 1 = 2$ and $x + 3 = 1 + 3 = 4$,

$x = 2$, $x + 1 = 2 + 1 = 3$ and $x + 3 = 2 + 3 = 5$,

$x = 3$, $x + 1 = 3 + 1 = 4$ and $x + 3 = 3 + 3 = 6$,

$x = 4$, $x + 1 = 4 + 1 = 5$ and $x + 3 = 4 + 3 = 7$,

$x = 5$, $x + 1 = 5 + 1 = 6$ and $x + 3 = 5 + 3 = 8$.

Hence, $R = \{(1, 3), (2, 4), (3, 5), (4, 6), (5, 7), (6, 8)\}$.

$\therefore$ Domain of R = {1, 2, 3, 4, 5, 6} and range of R = {3, 4, 5, 6, 7, 8}.

**Example 9.** *If* $R = \{(x, y) : x, y \in W, 2x + y = 8\}$, *then*

*(i) find the domain and the range of R.*

*(ii) write R as a set of ordered pairs.*

**Solution.** (*i*) Given $2x + y = 8$ and $x, y \in W$.

When $x = 0$, $2 \times 0 + y = 8 \Rightarrow y = 8$;

when $x = 1$, $2 \times 1 + y = 8 \Rightarrow y = 6$;

when $x = 2$, $2 \times 2 + y = 8 \Rightarrow y = 4$;

when $x = 3$, $2 \times 3 + y = 8 \Rightarrow y = 2$;

when $x = 4$, $2 \times 4 + y = 8 \Rightarrow y = 0$.

For all other values of $x \in \mathbf{W}$, we do not get $y \in \mathbf{W}$.

$\therefore$ Domain of R = {0, 1, 2, 3, 4} and range of R = {8, 6, 4, 2, 0}.

(*ii*) R as a set of ordered pairs can be written as

R = {(0, 8), (1, 6), (2, 4), (3, 2), (4, 0)}.

**Example 10.** *If* $R = \{(x, y) : x, y \in \mathbf{N}, x + 2y = 21\}$, then

(*i*) *find the domain and the range of R.*

(*ii*) *write R in the roster form.*

**Solution.** (*i*) Given $x + 2y = 21 \Rightarrow x = 21 - 2y$, $x, y \in \mathbf{N}$.

When $y = 1$, $x = 21 - 2 \times 1 = 19$;

when $y = 2$, $x = 21 - 2 \times 2 = 17$;

when $y = 3$, $x = 21 - 2 \times 3 = 15$;

... ... ...

... ... ...

when $y = 10$, $x = 21 - 2 \times 10 = 1$.

For all other values of $y \in \mathbf{N}$, we do not get $x \in \mathbf{N}$.

Domain of R = $\{x : (x, y) \in R$ for some $y \in \mathbf{N}\}$

= {19, 17, 15, ..., 1} and

range of R = $\{y : (x, y) \in R$ for some $x \in \mathbf{N}\}$

= {1, 2, 3, ..., 10}.

(*ii*) R in the roster form

= {(19, 1), (17, 2), (15, 3), ..., (10, 1)}.

**Example 11.** (*i*) *If* $R = \{(x, y) : x, y \in \mathbf{W}, x^2 + y^2 = 25\}$, *then find the domain and the range of R. Also write R in roster form.*

(*ii*) *If* $R = \{(x, y) : x, y \in \mathbf{Z}, x^2 + y^2 = 64\}$, *then write R in roster form.*

**Solution.** (*i*) $R = \{(x, y) : x, y \in \mathbf{W}, x^2 + y^2 = 25\}$.

When $x = 0$, $y = 5$; when $x = 3$, $y = 4$;

when $x = 4$, $y = 3$; when $x = 5$, $y = 0$.

For all other values of $x \in \mathbf{W}$, we do not get $y \in \mathbf{W}$.

$\therefore$ Domain of R = {0, 3, 4, 5} and range of R = {5, 4, 3, 0}.

$\therefore$ R = {(0, 5), (3, 4), (4, 3), (5, 0)}

(*ii*) $R = \{(x, y) : x, y \in \mathbf{Z}, x^2 + y^2 = 64\}$.

When $x = 0$, $y = 8, -8$;

when $x = 8$, $y = 0$; when $x = -8$, $y = 0$.

For all other values of $x \in \mathbf{Z}$, we do not get $y \in \mathbf{Z}$.

R = {(0, 8), (0, −8), (8, 0), (−8, 0)}

**Example 12.** *If* $R_1 = \{(x, y) : y = 2x + 7$, *where* $x \in \mathbf{R}$ *and* $-5 \le x \le 5\}$, *then find the domain and range of* $R_1$.

**Solution.** Given $R_1 = \{(x, y) : y = 2x + 7$, where $x \in \mathbf{R}$ and $-5 \le x \le 5\}$.

Since $x \in \mathbf{R}$ and $-5 \le x \le 5$, so domain of $R_1 = [-5, 5]$.

$-5 \le x \le 5 \Rightarrow -10 \le 2x \le 10$

$\Rightarrow -10 + 7 \le 2x + 7 \le 10 + 7 \Rightarrow -3 \le 7y \le 17$

$\therefore$ Range of $R_1 = [-3, 17]$.

**Example 13.** *Find the domain and the range of the relation R given by*

$$R = \{(x, y) : y = x + \frac{6}{x}, \text{ where } x, y \in N \text{ and } x < 6\}$$

**Solution.** Given $y = x + \frac{6}{x}$, $x, y \in$ N and $x < 6$.

When $x = 1$, $y = 1 + \frac{6}{1} = 7$ and $7 \in$ N, so $(1, 7) \in$ R;

when $x = 2$, $y = 2 + \frac{6}{2} = 5$ and $5 \in$ N, so $(2, 5) \in$ R;

when $x = 3$, $y = 3 + \frac{6}{3} = 5$ and $5 \in$ N, so $(3, 5) \in$ R;

when $x = 4$, $y = 4 + \frac{6}{4} \notin$ N, and

when $x = 5$, $y = 5 + \frac{6}{5} \notin$ N

$\therefore$ R = {(1, 7), (2, 5), (3, 5)}.

Domain of R = {1, 2, 3} and range of R = {7, 5}.

**Example 14.** *Find the linear relation between the components of the ordered pairs of the relation R where R = {(2, 1), (4, 7), (1, −2), ...}.*

**Solution.** Given R = {(2, 1), (4, 7), (1, −2), ...}.

Let $y = ax + b$ be the linear relation between the components of R.

Since $(2, 1) \in$ R, $\therefore$ $y = ax + b \Rightarrow 1 = 2a + b$ ...(*i*)

Also $(4, 7) \in$ R, $\therefore$ $y = ax + b \Rightarrow 7 = 4a + b$ ...(*ii*)

Subtracting (*i*) from (*ii*), we get $2a = 6 \Rightarrow a = 3$.

Substituting $a = 3$ in (*i*), we get $1 = 6 + b \Rightarrow b = -5$.

Substituting these values of $a$ and $b$ in $y = ax + b$, we get $y = 3x - 5$, which is the required linear relation between the components of the given relation.

**Example 15.** *If A = {1, 2, 3, 5}, B = {4, 6, 9} and a relation R from A to B is defined by R = {(x, y) : the difference between x and y is odd, x ∈ A, y ∈ B}. Then*

*(i) write R in the roster form*

*(ii) represent R by an arrow diagram.*

**Solution.** (*i*) Given A = {1, 2, 3, 5}, B = {4, 6, 9} and R is relation from A to B given by

R = {(x, y) : $x - y$ is odd, $x \in$ A, $y \in$ B}, therefore

R = {(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)}

(*ii*) The relation R can be represented by the arrow diagram shown in fig. 2.8.

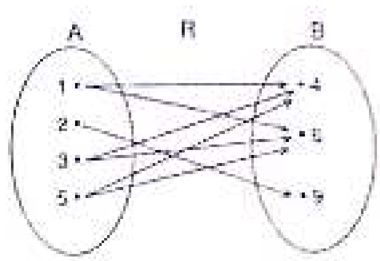



*Fig. 2.8.*

**Example 16.** *Let A = {3, 5} and B = {7, 11}. Let R = {(a, b) : a ∈ A, b ∈ B, a − b is odd}. Show that R is an empty relation from A to B.*

**Solution.** Here A × B = {(3, 7), (3, 11), (5, 7), (5, 11)}. Since none of the numbers 3 − 7, 3 − 11, 5 − 7, 5 − 11 is an odd number, therefore, none of pairs (3, 7), (3, 11), (5, 7) and (5, 11) belongs to R.

Hence, R is an empty relation.

**Example 17.** *Let R be the relation on* $\mathbb{Z}$ *defined by* $R = \{(x, y) : x, y \in \mathbb{Z}, x - y$ *is an odd ...*

*Find the domain and range of R.*

**Solution.** $R = \{(x, y) : x, y \in \mathbb{Z}, x - y$ is an odd integer$\}$.

Two cases arise :

**Case I.** If $x$ is an odd integer, take $y$ an even integer so that $x - y$ will be an odd integer.

**Case II.** If $x$ is an even integer, take $y$ an odd integer so that $x - y$ will be an odd integer.

Thus, on combining two cases, $x$ can be any integer. Also $y$ can be any integer.

Hence, domain of $R = \mathbb{Z}$ and range of $R = \mathbb{Z}$.

**Example 18.** *If* $A = \{2, 4, 6, 9\}$, $B = \{4, 6, 18, 27, 54\}$ *and a relation R from A to B is defined by* $R = \{(a, b) : a \in A, b \in B$, *a is a factor of b and* $a < b\}$, *then find R in roster form. Also find its domain and range.*

**Solution.** Given $A = \{2, 4, 6, 9\}$, $B = \{4, 6, 18, 27, 54\}$ and

$R = \{(a, b) : a \in A, b \in B$, $a$ is a factor of $b$ and $a < b\}$.

Since 2 is a factor of 4 and $2 < 4$, so $(2, 4) \in R$.

Similarly, $(2, 6), (2, 18), (2, 54) \in R$.

Also $(6, 18), (6, 54), (9, 18), (9, 27), (9, 54) \in R$.

$\therefore \quad R = \{(2, 4), (2, 6), (2, 18), (2, 54), (6, 18), (6, 54), (9, 18), (9, 27), (9, 54)\}$.

Domain of $R = \{2, 6, 9\}$, range of $R = \{4, 6, 18, 27, 54\}$.

## EXERCISE 2.2

*Very short answer type questions (1 to 10) :*

1. If A and B are two sets such that $n(A) = 2$ and $n(B) = 3$, find the number of relations from
   (*i*) A to B  (*ii*) B to A  (*iii*) A to A.
2. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find
   (*i*) $A \times B$  (*ii*) number of relations from A to B.
3. Let $A = \{1, 2\}$ and $B = \{x, y, z\}$, find the number of relations from
   (*i*) A to A  (*ii*) A to B  (*iii*) B to A  (*iv*) B to B.
4. If a relation $R = \{(-2, 1), (0, 2), (3, 1), (0, -1), (4, 2), (5, 1)\}$, then write its domain and range.
5. If $A = \{2, 3, 5\}$, $B = \{2, 4, 6\}$ and R is the relation from A to B defined by $R = \{(x, y) : x \in A, y \in B$ and $x < y\}$, then write R in the roster form.
6. If $A = \{1, 3, 5, 7, 8\}$ and $B = \{2, 3, 4, 6, 8, 10\}$ and R be the relation 'is one less than' from A to B, then write R in the roster form.
7. If $A = \{2, 3, 4\}$, $B = \{4, 6, 9, 10\}$ and $R = \{(x, y) : (x, y) \in A \times B$ such that $x$ is a factor of $y\}$, then write R in roster form.
8. If $A = \{2, 3, 4, 5, 6\}$ and R is a relation from A to A defined by $R = \{(x, y) : y = x + 1, x, y \in A\}$, then list the elements of R.
9. If $A = \{1, 2, 3, \ldots, 17\}$ and R is a relation on A defined by $R = \{(x, y) : 3x - y = 0, x, y \in A\}$, then write R in the roster form.
10. Write the following relations in the roster form:
    (*i*) $R = \{(x, x^3) : x$ is a prime number less than $10\}$
    (*ii*) $R = \{(x - 2, x^2) : x$ is a prime number less than $10\}$.
11. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Let R be a relation from A to B defined by $R = \{(1, x), (1, z), (3, x), (4, y)\}$.
    (*i*) Find the domain and range of R.  (*ii*) Represent R by an arrow diagram.

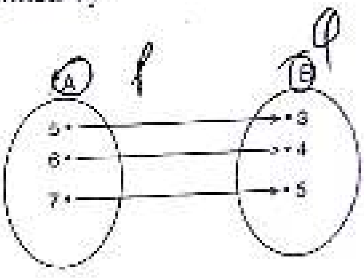
12. Let a relation R = {(1, −1), (2, 0), (3, 1), (4, 2), (5, 3)}, then
 (*i*) write the domain and the range of R. (*ii*) write R in the builder form.

13. Let A = {2, 4, 6}, B = {4, 6, 18} and R be the relation '*is a factor of*' from A to B. Find R as a set of ordered pairs and represent it by an arrow diagram.

14. Given R = {(x, y) ; y = x − 3, x, y ∈ Z}. State which of the ordered pairs belong to the given relation :
 (*i*) (5, 2) (*ii*) (1, 2) (*iii*) (0, −3) (*iv*) (7, −4) (*v*) (−4, 1).

15. Determine the domain and the range of the relation R defined by
 R = {(x, x − 5) : x ∈ {0, 1, 2, 3, 4, 5}}.

16. The adjoining diagram shows a relationship between the sets P and Q. Write this relation in
 (*i*) roster form
 (*ii*) set builder form.
 What is its domain and range?




17. Determine the domain and the range of the following relations :
 (*i*) {(x, y) : x ∈ N, y ∈ N and x + y = 10}
 (*ii*) {(x, y) : x ∈ N, x < 5, y = 3}.

18. Let R be the relation on N defined by
 R = {(a, b) : a ∈ N, b ∈ N and a + 3b = 12}, then
 (*i*) list the elements of R
 (*ii*) find the domain of R
 (*iii*) find the range of R.

19. If R = {(x, y) : x, y ∈ W, $x^2 + y^2 = 100$}, then find the domain and the range of R. Also write R in roster form.

20. If R is the relation on N defined by
 R = {(x, y) : $y = x - \frac{12}{x}$, x, y ∈ N}, then find
 (*i*) R in roster form (*ii*) domain of R (*iii*) range of R.

21. If A = {2, 3, 6, 10} and B = {1, 6, 10}, find the elements of the subset of A × B corresponding to the relation R 'is less than'. Also represent this relation by an arrow diagram.

22. Write down the domain and the range of the relation (x, y) : x = 3y and x and y are natural numbers less than 10.

23. Let A = {−2, −1, 0, 1, 2}, list the ordered pairs satisfying each of the following relations on A :
 (*i*) '*is greater than*'. (*ii*) '*is the square of*'. (*iii*) '*is the negative of*'.

24. If A = {1, 3, 5, 6} and B = {3, 4, 5}, write the relation R as a set of ordered pairs if
 (*i*) R = {(x, y) : (x, y) ∈ A × B : x − y is even}
 (*ii*) R = {(x, y) : (x, y) ∈ A × B : xy is odd}.

25. Let R = {(x, y) : x, y ∈ Z, y = 2x − 4}. If (a, −2) and (4, $b^2$) belong to R, find the values of a and b.

26. Find the linear relation between the components of the ordered pairs of the relation R where
 (*i*) R = {(−1, −1), (0, 2), (1, 5), ...}. (*ii*) R = {(0, 2), (−1, 5), (2, −4), ...}.

## 2.4 FUNCTIONS

*A function is a special case of a relation.* To be specific, let X, Y be two non-empty sets and R (or *f*) be a relation from X to Y, then R may not relate an element of X to an element of Y or it may relate an element of X to more than one element of Y. But a function relates each element of X to a unique element of Y. We have :

**Definition.** *If X, Y are two non-empty sets then a subset f of X × Y is called a* ***function*** *(or map) from X to Y iff for each x ∈ X, there exists a unique y ∈ Y such that (x, y) ∈ f. It is written as f : X → Y.*

*Thus, a subset f of X × Y is called a function from X to Y iff*

*(i) for each x ∈ X, there exists y ∈ Y such that (x, y) ∈ f and*

*(ii) no two different ordered pairs have the same first component.*

***In other words, a function from X to Y is a rule (or correspondence) which associates to each element x of X, a unique element y of Y.***

A function '$f$' can be thought of as a mechanism (or device) which gives a unique output $f(x)$ to every input $x$.

*Fig. 2.9.*

**Image of an element.** *The unique element y ∈ Y is called the* ***image*** *of the element x of X under the function f : X → Y. It is denoted by f (x) i.e. y = f (x). The element y is also called the* ***value*** *of the function f at x.*

## 2.4.1 Domain and range of a function

*Let f be a function from X to Y, then the set X is called the* ***domain*** *of the function f and the set Y is called the* ***codomain.***

*The set consisting of all the images of the elements of X under the function f is called the* ***range*** *of f. It is denoted by f (X).* Thus,

*range of* $f = \{f(x) : \text{for all } x \in X\}$.

Note that range of $f$ is a subset of Y (codomain) which may or may not be equal to Y.

***For example :***

(1) Let X = {1, 2, 3, 4, 5}, Y = {0, 1, 2, 3, 5, 7, 9, 11, 13} and

(*i*) $f$ = {(1, 1), (2, 0), (3, 7), (4, 9), (5, 13)}, then $f$ is a function from X to Y because each element of X has a unique image in Y.

Range of $f$ = {1, 0, 7, 9, 13}.

Note that some elements of Y are not associated with any element of X.

(*ii*) $f$ = {(1, 3), (2, 3), (3, 5), (4, 7), (5, 5)}, then $f$ is a function from X to Y because each element of X has a unique image in Y.

Range of $f$ = {3, 5, 7}.

Note that the second components may repeat.

(*iii*) $f$ = {(1, 5), (2, 7), (4, 9), (5, 0)}, then $f$ is not a function from X to Y because the element 3 of X has no image in Y.

(*iv*) $f$ = {(1, 1), (1, 2), (2, 3), (3, 5), (4, 7), (5, 11)}, then $f$ is not a function because the different pairs (1, 1) and (1, 2) have same first component *i.e.* the element 1 of X has two different images.

(2) Let X = {*a*, *b*, *c*, *d*} and Y = {*p*, *q*, *r*, *s*, *t*}, then

(*i*) the rule depicted by the adjoining arrow diagram represents a function from X to Y because each element of X has a unique image in Y.

Range of the function = {*p*, *q*, *r*, *t*}.

Note that the element *s* of Y is not associated with any element of X.

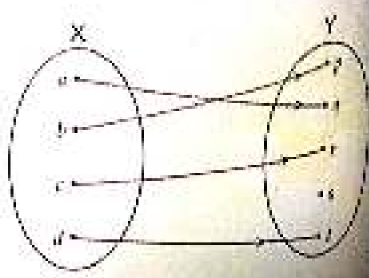


*Fig. 2.10.*

(*ii*) the rule depicted by the adjoining arrow diagram represents a function from X to Y because each element of X has a unique image in Y.

Range of the function = {*p*, *r*, *s*}.

Note that the elements *a* and *d* of X have the same image *s* in Y.

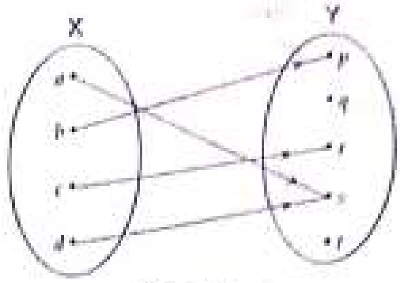



Fig. 2.11.

(*iii*) the rule depicted by the adjoining arrow diagram does not represent a function from X to Y because the element *a* of X has two different images *p* and *r* in Y.

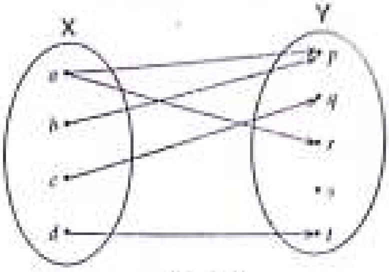



Fig. 2.12.

(*iv*) the rule depicted by the adjoining arrow diagram does not represent a function from X to Y because the element *c* of X has no image in Y.

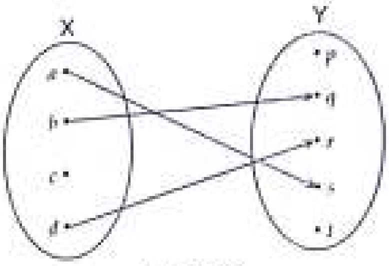



Fig. 2.13.

### 2.4.2 Main features of a function

Let *f* be a function from X to Y, then

(*i*) to every $x \in X$, there exists a unique element $y \in Y$ such that $y = f(x)$.

(*ii*) no element of X can have more than one images in Y.

(*iii*) there may be elements of Y which are not associated with any element of X.

(*iv*) distinct elements of X may have same image in Y.

(*v*) function *f* is determined when $f(x)$ is known for all $x \in X$.

### 2.4.3 Types of functions

1. **One-one function.** *A function 'f' from X to Y is called* ***one-one*** *(or* ***injective****) iff different elements of X have different images in Y i.e. iff* $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ *for all* $x_1, x_2 \in X$, *or equivalently,* $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ *for all* $x_1, x_2 \in X$.

2. **Many-one function.** *A function 'f' from X to Y is called* ***many-one*** *iff two or more elements of X have same image in Y.* In other words, a function '*f*' from X to Y is called *many-one* iff it is not one-one.

3. **Onto function.** *A function 'f' from X to Y is called* ***onto*** *(or* ***surjective****) iff each element of Y is the image of atleast one element of X i.e. iff codomain of f = range of f i.e. iff* $Y = f(X)$.

4. **Into function.** *A function 'f' from X to Y is called* ***into*** *iff there exists atleast one element in Y which is not the image of any element of X i.e. iff range of f is a proper subset of codomain of f.* In other words, a function '*f*' from X to Y is called *into* iff it is not onto.

5. **One-one correspondence.** *A function 'f' from X to Y is called a* ***one-one correspondence*** *(or* ***bijective****) iff f is both one-one and onto.*

Since we will often be proving that certain functions are one-one or onto or both, so we outline the techniques to be used :

(i) *To prove that f is one-one, we must show that either* $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ *for all* $x_1, x_2 \in X$ *or for all* $x_1 \neq x_2 \in X \Rightarrow f(x_1) \neq f(x_2)$.

(ii) *To prove that f is onto, we must show that either for every* $y \in Y$, *there exists atleast one element* $x \in X$ *such that* $y = f(x)$ *or* $f(X) = Y$.

*For example :*

1. Let X = {a, b, c, d}, Y = {p, q, r, s} and '*f*' be the function from X to Y depicted by the adjoining arrow diagram, then *f* is

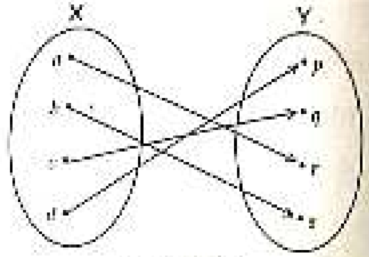


*Fig.* 2.14.

(i) a *one-one function.*

*Reason* : different elements of X have different images in Y.

(ii) an *onto function.*

*Reason* : each element of Y is the image of an element of X or range of *f* = {p, q, r, s} = codomain of *f*.

Thus, the function '*f*' from X to Y is a *one-one onto function i.e. f is a one-one correspondence* from X to Y.

Note that n(X) = 4 = n(Y).

2. Let X = {a, b, c}, Y = {p, q, r, s} and '*f*' be the function from X to Y depicted by the adjoining arrow diagram, then *f* is

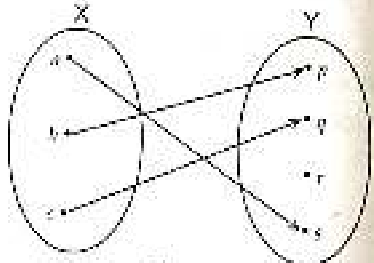


*Fig.* 2.15.

(i) a *one-one function.*

*Reason* : different elements of X have different imges in Y.

(ii) an *into function.*

*Reason* : there exists $r \in Y$ which is not the image of any element of X or range of *f* = {p, q, s} which is a proper subset of Y.

Thus, the function *f* from X to Y is a *one-one into function.*

3. Let X = {–1, 2, 5, 8, 11}, Y = {4, 6, 8} and '*f*' be the function from X to Y depicted by the adjoining arrow diagram, then *f* is

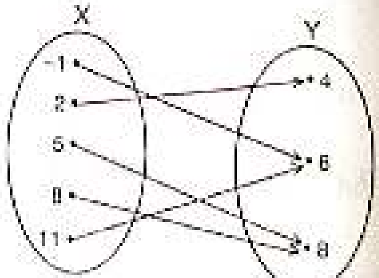


*Fig.* 2.16.

(i) a *many-one function.*

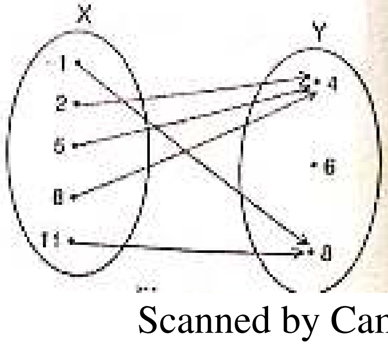
*Reason* : different elements –1 and 11 of X have the same image 6 in Y.

(ii) an *onto function.*

*Reason* : each element of Y is the image of atleast one element of X.

Thus, the function *f* from X to Y is a *many-one onto function.*

4. Let X = {–1, 2, 5, 8, 11}, Y = {4, 6, 8} and '*f*' be the function from X to Y depicted by the adjoining arrow diagram, then *f* is




(i) a *many-one function.*

*Reason* : different elements 2, 5 and 8 of X have the same image 4 in Y.

(*ii*) an *into function*.

*Reason* : there exists 6 ∈ Y which is not the image of any element of X.

Thus, the function *f* from X to Y is a *many-one into function*.

5. Let X = {–1, 0, 1, 3, 5}, Y = {–2, 0, 2, 6, 10}, then the function *f* defined by $f(x) = 2x$ for all $x \in X$ is both one-one and onto *i.e.* *f* is a one-one correspondence.

It is depicted by fig. 2.18. Note that

$n(X) = 5 = n(Y)$.

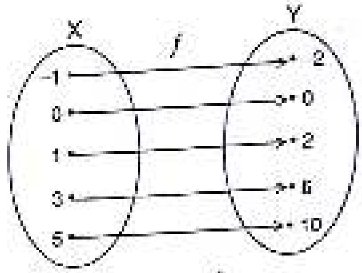



*Fig.* 2.18.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Which of the following relations are functions? Give reasons.*

(*i*) R = {(2, 1), (3, 1), (4, 2)}

(*ii*) R = {(2, 3), ($\frac{1}{2}$, 0), (2, 7), (–4, 6)}

(*iii*) R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7)}.

**Solution.** (*i*) Domain of R = {2, 3, 4}. We note that each element of the domain of R has a unique image, therefore, the relation R is a function.

(*ii*) Domain of R = {2, $\frac{1}{2}$, –4}. We note that the element 2 of the domain of R has two different images 3 and 7, therefore, the relation R is not a function.

(*iii*) Domain of R = {1, 2, 3, 4, 5, 6}. We note that each element of the domain of R has a unique image, therefore, the relation R is a function.

**Example 2.** *If A = {1, 2, 3} and f, g, h and s are relations corresponding to the subsets of A × A indicated against them, which of f, g, h and s are functions ? In case of a function, find its domain and range.*

(*i*) *f* = {(2, 1), (3, 3)}

(*ii*) *g* = {(1, 2), (1, 3), (2, 3), (3, 1)}

(*iii*) *h* = {(1, 3), (2, 1), (3, 2)}

(*iv*) *s* = {(1, 2), (2, 2), (3, 1)}.

**Solution.** (*i*) *f* is not a function because the element 1 of A does not appear as the first component of ordered pairs of *f*, so 1 has no image in A.

(*ii*) *g* is not a function because the different pairs (1, 2) and (1, 3) of *g* have same first component *i.e.* the element 1 of A has two different images in A.

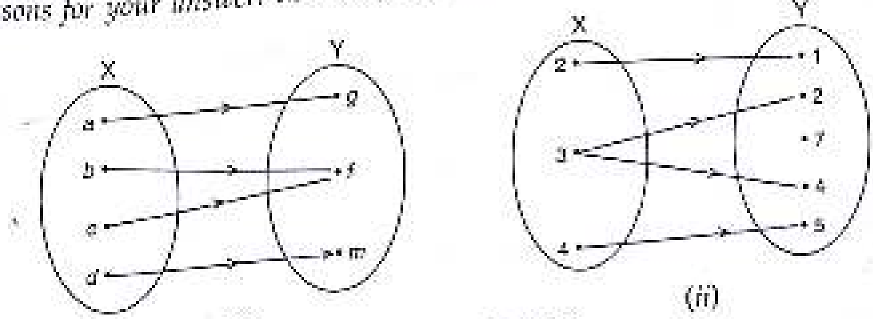
(*iii*) *h* is a function because each element of A has a unique image in A.

Domain of *h* = {1, 2, 3} = A and range of *h* = {3, 1, 2} = A.

(*iv*) *s* is a function because each element of A has a unique image in A.

Domain of *s* = {1, 2, 3} = A and range of *s* = {2, 1}.

**Example 3.** *Consider the following diagrams carefully and state whether they represent functions. Give reasons for your answer. In case of a function, write its domain and range.*




(*i*) Fig. 2.19. (*ii*)

**Solution.** (*i*) The given diagram represents a function because each element of the set $\{a, b, c, d\}$ has a unique image in the set $\{g, f, m\}$.

Its domain = $\{a, b, c, d\}$ and range = $\{g, f, m\}$.

(*ii*) The given diagram does not represent a function because the element 3 of the set $\{2, 3, 4\}$ has two different images 2 and 4 in the set $\{1, 2, 4, 5, 7\}$.

**Example 4.** *Is the given relation a function? Justify your answer.*

(*i*) $f = \{(x, |x|) : x \in R\}$ (*ii*) $g = \left\{\left(n, \frac{1}{n}\right) : n \text{ is a positive integer}\right\}$

**Solution.** (*i*) Given $f = \{(x, |x|) : x \in \mathbf{R}\}$.

For every $x \in \mathbf{R}$, there is a unique image as $|x| \in \mathbf{R}$.

Therefore, $f$ is a function.

(*ii*) Given $g = \left\{\left(n, \frac{1}{n}\right) : n \text{ is a positive integer}\right\}$.

For every positive integer $n$, there is a unique image as $\frac{1}{n} \in \mathbf{R}$.

Therefore, $g$ is a function.

**Example 5.** *Let N be the set of natural numbers and the relation R be defined on N by*

$R = \{(x, y) : y = 2x, x, y \in N\}$.

*What is the domain, codomain and range of R? Is this relation a function?*

**Solution.** Given $R = \{(x, y) : y = 2x, x, y \in \mathbf{N}\}$.

$\therefore$ Domain of R = **N**, codomain of R = **N**

and range of R is the set of even natural numbers.

Since every natural number $x$ has a unique image $2x$, therefore, the relation R is a function.

**Example 6.** *A relation 'f' is defined by $f : x \to x^2 - 2$, where $x \in \{-1, -2, 0, 2\}$.*

(*i*) *List the elements of f.* (*ii*) *Is f a function?*

**Solution.** Relation $f$ is defined by $f : x \to x^2 - 2$ *i.e.* $f(x) = x^2 - 2$, where $x \in \{-1, -2, 0, 2\}$

(*i*) $f(-1) = (-1)^2 - 2 = 1 - 2 = -1$,

$f(-2) = (-2)^2 - 2 = 4 - 2 = 2$,

$f(0) = 0^2 - 2 = 0 - 2 = -2$,

$f(2) = 2^2 - 2 = 4 - 2 = 2$.

$\therefore f = \{(-1, -1), (-2, 2), (0, -2), (2, 2)\}$

(*ii*) We note that each element of the domain of $f$ has a unique image, therefore, the relation $f$ is a function.

**Example 7.** *Let $A = \{9, 10, 11, 12, 13\}$ and let $f : A \to N$ be defined by $f(n)$ = the highest prime factor of $n$, $n \in A$. Write $f$ as a set of ordered pairs and find the range of $f$.*

**Solution.** Given A = {9, 10, 11, 12, 13}. We express each element of A as the product of its prime factors.

$9 = 3 \times 3$, $10 = 2 \times 5$, 11 is prime, $12 = 2 \times 2 \times 3$ and 13 is prime.

Given $f : A \to N$ by $f(x)$ = highest prime factor of $n$, $n \in A$.

$\therefore f(9) = 3, f(10) = 5, f(11) = 11, f(12) = 3$ and $f(13) = 13$.

$\therefore f = \{(9, 3), (10, 5), (11, 11), (12, 3), (13, 13)\}$.

**Example 8.** *Let f be the subset of* $\mathbf{Z} \times \mathbf{Z}$ *defined by* $f = \{(ab, a + b) : a, b \in \mathbf{Z}\}$.
*Is f a function from* **Z** *to* **Z**? *Justify your answer.*

**Solution.** Given $f = \{(ab, a + b) : a, b \in \mathbf{Z}\}$ i.e. $f(ab) = a + b$, $a, b \in \mathbf{Z}$.

Let $a = 0$, $b = 2$, then $f(0 \times 2) = 0 + 2 \Rightarrow f(0) = 2$.

Again, let $a = 0$, $b = 3$, then $f(0 \times 3) = 0 + 3 \Rightarrow f(0) = 3$.

Thus, the element 0 of the domain of $f$ has two different images, therefore, the relation $f$ is not a function.

**Example 9.** *Let* $f = \{(1, 1), (2, 3), (0, -1), (-1, -3), \ldots\}$ *be a function from* **Z** *to* **Z** *defined by* $f(x) = ax + b$, *for some integers a and b. Determine a and b.*

**Solution.** Given $f(x) = ax + b$.

Since $(1, 1) \in f$, $f(1) = 1 \Rightarrow a + b = 1$ ...(i)

$(2, 3) \in f$, $f(2) = 3 \Rightarrow 2a + b = 3$ ...(ii)

Subtracting (i) from (ii), we get $a = 2$.

Substituting $a = 2$ in (i), we get $2 + b = 1 \Rightarrow b = -1$.

Hence, $a = 2$, $b = -1$.

**Example 10.** *If A and B are finite sets such that* $n(A) = p$ *and* $n(B) = q$, *then find the number of functions from A to B.*

**Solution.** Any element of set A, say $x_i$ $(1 \le i \le p)$ can be connected with an element of set B in $q$ ways. Similarly, any other element of set A can also be connected with an element of set B in $q$ ways.

Thus, every element of set A can be connected with an element of set B in $q$ ways.

Therefore, the total number of functions from set A to set B

$= q \times q \times q \ldots p$ times $= q^p$.

**Example 11.** *Show that the function* $f : \mathbf{N} \to \mathbf{N}$, *defined by* $f(x) = 2x$, *is one-one but not onto.*

**Solution.** *The function f is one-one.*

*Reason* : Let $x_1, x_2 \in \mathbf{N}$ be such that $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one.

*The function f is not onto.*

*Reason* : As $1 \in \mathbf{N}$ (codomain of $f$) and there does not exist any $x \in \mathbf{N}$ (domain of $f$) such that $f(x) = 1$. So, $f$ is not onto.

**Example 12.** *Prove that the function* $f : \mathbf{R} \to \mathbf{R}$, *defined by* $f(x) = 2x$, *is one-one and onto.*

**Solution.** *The function f is one-one.*

*Reason* : Let $x_1, x_2 \in \mathbf{R}$ be such that $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_1 \Rightarrow f$ is one-one.

*The function f is onto.*

*Reason* : Consider any $y \in \mathbf{R}$ (codomain of $f$).

Certainly, $x = \frac{y}{2} \in \mathbf{R}$ (domain of $f$).

Thus, for all $y \in \mathbf{R}$ (codomain of $f$), there exists $x = \frac{y}{2} \in \mathbf{R}$ (domain of $f$) such that

$f(x) = f\left(\frac{y}{2}\right) = 2 \cdot \frac{y}{2} = y \Rightarrow f$ is onto.

## EXERCISE 2.3

***Very short answer type questions (1 to 7) :***

1. Which of the following relations are functions ? Give reasons. If it is a function, determi[ne] its domain and range.

   (*i*) {(1, 1), (1, 2), (1, 3), (1, 4)}  (*ii*) {(*a*, *b*), (*b*, *c*), (*c*, *d*), (*d*, *e*)}

   (*iii*) {(1, 2), (3, 1), (1, 3), (4, 1)}  (*iv*) {(1, 2), (2, 2), (3, 2), (4, 2)}

   (*v*) {(2, 1), (0, – 1), (3, 1), (5, 4), (–1, 0), (3, 4), (1, 0)}

   (*vi*) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 3), (14, 7)}.

2. If X = {–4, 1, 2, 3} and Y = {*a*, *b*, *c*}, which of the following relations is a function fro[m] X to Y ?

   (*i*) {(–4, *a*), (1, *a*), (2, *b*)}  (*ii*) {(–4, *b*), (1, *b*), (2, *a*), (3, *c*)}

   (*iii*) {(–4, *a*), (1, *a*), (2, *b*), (3, *c*), (1, *b*)}.

3. Let A = {1, 2, 3, 4}, B = {1, 5, 9, 11, 15, 16} and $f$ = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)}. Are the following true?

   (*i*) $f$ is a relation from A to B.

   (*ii*) $f$ is a function from A to B?

   Justify your answer in each case.

4. Is the given relation a function? Justify your answer.

   (*i*) $f = \{(x, x) : x \in R\}$

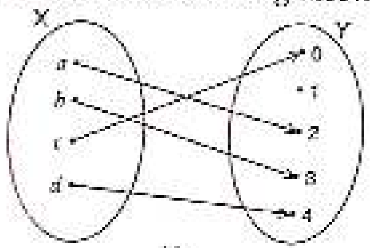
   (*ii*) $g = \{(n, n^2) : n$ is a positive integer$\}$

   (*iii*) $h = \{(x, 3) : x \in R\}$

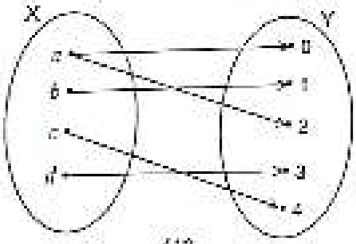
5. (*i*) A function '$f$' is defined by $f(x) = x^2 + 1$ where $x \in \{-1, 0, 1, 3\}$. List the elements of $f$.

   (*ii*) If a function $f$ is defined by $f(x) = 2x - 1$ where $x \in \{-2, 0, 3, 5\}$, then find its range.

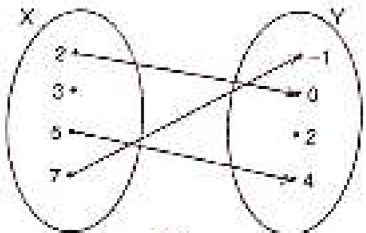
6. Which of the following arrow diagrams represent a function?



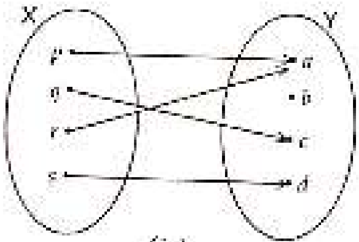

(*i*)




(*ii*)

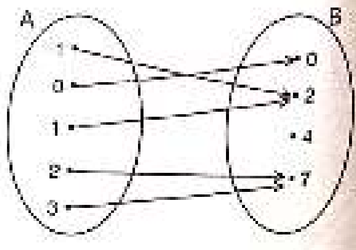



(*iii*)

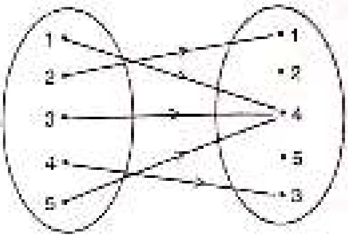



(*iv*)

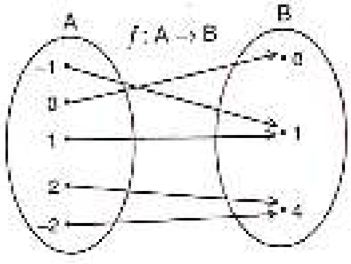
7. Does the adjacent arrow diagram represent a function? If so, write its range.



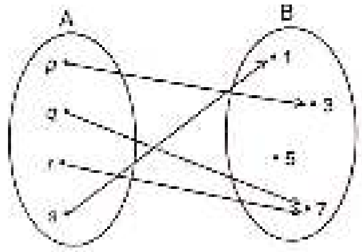
8. The adjacent arrow diagram represents a relation. Represent the relation in roster form. Is this relation a function ? Give reasons for your answer.




9. The adjacent arrow diagram represents a relation. List the pairs that satisfy the relation. Is this relation a function? Also find the rule (relation) for the above correspondence.




10. Write the relation represented by the adjoining diagram, by listing the ordered pairs. State the domain, the codomain and the range of the relation. Is the relation a function?




11. $A = \{-2, -1, 1, 2\}$ and $f = \left\{\left(x, \frac{1}{x}\right) : x \in A\right\}$.

(i) List the domain of $f$. (ii) List the range of $f$. (iii) Is $f$ a function?

12. Express the following function as a set of ordered pairs and find its range :

$f : X \to R$ defined by $f(x) = x^3 + 1$, where $X = \{-1, 0, 3, 9, 7\}$.

13. If $f(x) = ax + b$, where $a$ and $b$ are integers, $f(-1) = -5$ and $f(3) = 3$, find $a$ and $b$.

14. If $(a, 8)$ and $(2, b)$ are ordered pairs which belong to the mapping $f : x \to 3x + 4$ where $x \in R$, find $a$ and $b$.

15. If a function $f$ from R to R is defined by $f = \{(x, 3x - 5) : x \in R\}$, find the values of $a$ and $b$ given that $(a, 4)$ and $(1, b)$ belong to $f$.

16. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and $f : \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that $f$ is one-one but not onto.

17. Show that the function $f : Q \to Q$ defined by $f(x) = 3x - 2$ is one-one.

18. Show that the function $f : N \to N$ defined by $f(x) = 2x - 1$ is one-one but not onto.

19. Show that the function $f : N \to N$ defined by $f(n) = n^2$ is injective.

20. Show that the function $f : N \to N$ defined by $f(m) = m^2 + m + 3$ is one-one.

21. In each of the following cases, state whether the function is bijective or not. Justify your answer.

(i) $f : R \to R$ defined by $f(x) = 2x - 1$

(ii) $f : R \to R$ defined by $f(x) = 3 - 4x$

## 2.5 REAL FUNCTIONS

A function which has either R or one of its subsets as its range is called a **real valued function.** Further, if its domain is also either R or a subset of R, then it is called a **real function.** Thus, a real function is defined as :

*Let R be the set of all real numbers and X, Y be two non-empty subsets of R, then a rule (or correspondence) 'f' which associates to each $x \in X$, a unique element y of Y is called a real valued function of the real variable or simply a real function, and we write it as $f : X \to Y$.*

Most of the time, we shall be given a function in the shape of a 'formula' without mentioning its domain. In such cases, the domain will be the *largest admissible subset of R on which the given formula is meaningful*. Keeping this in view i.e. when the domain of a function is not given, a real function can be redefined as :

*A real function 'f' is a rule(s) which associates to each possible real number x, a unique real number f(x).*

## 2.5.1 Domain and range of a real function

*Let 'f' be a real function, then :*

*The set of all possible real numbers x for which f(x) is a real number is called the domain of the function f.* It is usually denoted by $D_f$. Thus,

$D_f = \{x : x \in R, f(x) \in R\}$.

*The set of images f(x) for all $x \in D_f$ is called the range of f, it is usually denoted by $R_f$.* Thus,

$R_f = \{f(x) ; \text{ for all } x \in D_f\}$.

**REMARKS**

1. A function does not exist if its domain is empty set.
2. If the domain of a real valued function is given, then we are not to find the admissible subset of R.
3. The distinctive property of a function is that, for a given element x of the domain, f(x) is uniquely determined element of codomain.

## 2.5.2 Equal functions

*Two functions f and g are called equal, written as f = g, if and only if*

*(i) domain of f = domain of g and*

*(ii) f(x) = g (x) for all x in domain of f (or g).*

Thus, $f \neq g$ iff either $D_f \neq D_g$ or there exists atleast one real number $x \in D_f$ such that $f(x) \neq g(x)$.

**NOTES**

1. If a and b are real numbers such that a < b, then
   - (i) $(x - a)(x - b) < 0$ iff $a < x < b$ i.e. iff $x \in (a, b)$.
   - (ii) $(x - a)(x - b) \leq 0$ iff $a \leq x \leq b$ i.e. iff $x \in [a, b]$.
   - (iii) $(x - a)(x - b) > 0$ iff either $x < a$ or $x > b$ i.e. iff $x \in (-\infty, a) \cup (b, \infty)$.
   - (iv) $(x - a)(x - b) \geq 0$ iff either $x \leq a$ or $x \geq b$ i.e. iff $x \in (-\infty, a] \cup [b, \infty)$.
2. It may be noted that $\frac{f(x)}{g(x)}$ has same sign as that of $f(x) \cdot g(x)$ because $g(x) \neq 0$ and $[g(x)]^2 > 0$ for all $x \in R$, and the multiplication by a positive number does not change the sign (symbol) of an inequality.
3. (i) If x is a real number, then $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

   (ii) $|x| \geq 0$ for all real numbers x.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *A real function f is defined by f(x) = 2x − 5. Find the values of :*

*(i) f(0)* *(ii) f(7)* *(iii) f(−3).*

**Solution.** Given $f(x) = 2x - 5$.

(i) $f(0) = 2 \times 0 - 5 = -5$

(ii) $f(7) = 2 \times 7 - 5 = 9$

(iii) $f(-3) = 2 \times (-3) - 5 = -11$.

**Example 2.** *Let 'f' be a function defined by $f: x \to 5x^2 + 2, x \in R$.*

*(i) Find the image of 3 under f.* *(ii) Find $f(3) \times f(2)$.*

*(iii) Find x such that $f(x) = 22$.*

**Solution.** Given $f(x) = 5x^2 + 2, x \in \mathbf{R}$.

(i) $f(3) = 5 \times 3^2 + 2 = 5 \times 9 + 2 = 47$.

(ii) $f(2) = 5 \times 2^2 + 2 = 5 \times 4 + 2 = 22$,

$\therefore f(3) \times f(2) = 47 \times 22 = 1034$.

(iii) $f(x) = 22 \Rightarrow 5x^2 + 2 = 22$

$\Rightarrow 5x^2 = 20 \Rightarrow x^2 = 4$

$\Rightarrow x = 2, -2$.

**Example 3.** *Find the domain of the following functions :*

(i) $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$ (ii) $f(x) = \dfrac{x+7}{x^2-8x+4}$.

**Solution.** (i) Given $f(x) = \dfrac{x^2+2x+1}{x^2-8x+12}$.

*For* $D_f$, $f(x)$ must be a real number

$\Rightarrow \dfrac{x^2+2x+1}{x^2-8x+12}$ must be a real number

$\Rightarrow x^2 - 8x + 12 \neq 0 \Rightarrow (x-2)(x-6) \neq 0$

$\Rightarrow x \neq 2, 6 \Rightarrow D_f = \mathbf{R} - \{2, 6\}$.

(ii) Given $f(x) = \dfrac{x+7}{x^2-8x+4}$.

*For* $D_f$, $f(x)$ must be a real number

$\Rightarrow \dfrac{x+7}{x^2-8x+4}$ must be a real number

$\Rightarrow x^2 - 8x + 4 \neq 0 \Rightarrow x \neq \dfrac{8 \pm \sqrt{64 - 4 \times 1 \times 4}}{2} \Rightarrow x \neq 4 \pm 2\sqrt{3}$

$\Rightarrow D_f = \mathbf{R} - \{4 \pm 2\sqrt{3}\}$.

**Example 4.** *If* $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$, *find* $f(-2) + f\left(\dfrac{1}{3}\right)$.

**Solution.** Given $f(x) = \dfrac{x^2 - 3x + 1}{x - 1}$, $D_f = \mathbf{R} - \{1\}$.

$$\therefore \quad f(-2) = \frac{(-2)^2 - 3(-2) + 1}{-2-1} = \frac{4+6+1}{-3} = -\frac{11}{3} \text{ and}$$

$$f\left(\frac{1}{3}\right) = \frac{\left(\frac{1}{3}\right)^2 - 3 \times \frac{1}{3} + 1}{\frac{1}{3} - 1} = \frac{\frac{1}{9} - 1 + 1}{-\frac{2}{3}} = \frac{\frac{1}{9}}{-\frac{2}{3}}$$

$$= \frac{1}{9} \times \left(-\frac{3}{2}\right) = -\frac{1}{6}.$$

$$\therefore \quad f(-2) + f\left(\frac{1}{3}\right) = -\frac{11}{3} - \frac{1}{6} = \frac{-22-1}{6} = -\frac{23}{6} = -3\frac{5}{6}.$$

**Example 5.** *If $f(x) = x^2 - 3x + 1$, find $x \in R$ such that $f(2x) = f(x)$.*

**Solution.** Given $f(x) = x^2 - 3x + 1$, $D_f = R$.

$\therefore\ f(2x) = (2x)^2 - 3(2x) + 1 = 4x^2 - 6x + 1$.

As $f(2x) = f(x)$ (given)

$\Rightarrow\ 4x^2 - 6x + 1 = x^2 - 3x + 1$

$\Rightarrow\ 3x^2 - 3x = 0 \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$

$\Rightarrow\ x = 0, 1$.

**Example 6.** *Find the range of the function $f(x) = 2 - 3x$, $x \in R$, $x > 0$.*

**Solution.** Given $f(x) = 2 - 3x$, $x \in R$, $x > 0$.

For $R_f$, let $y = f(x) = 2 - 3x$, $x \in R$, $x > 0$.

Now, $x > 0 \Rightarrow 3x > 0 \Rightarrow -3x < 0$

$\Rightarrow\ 2 - 3x < 2 \Rightarrow y < 2$

$\Rightarrow\ R_f = (-\infty, 2)$.

**Example 7.** *Find the domain and the range of the function $f(x) = 3x^2 - 5$. Also find $f(-3)$ and the numbers which are associated with the number 43 in its range.*

**Solution.** Given $f(x) = 3x^2 - 5$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow\ 3x^2 - 5$ must be a real number, which is a real number for every $x \in R$

$\Rightarrow\ D_f = R$.

For $R_f$, let $y = f(x) = 3x^2 - 5$ ...(i)

We know that for all $x \in R$, $x^2 \geq 0 \Rightarrow 3x^2 \geq 0$

$\Rightarrow\ 3x^2 - 5 \geq -5 \Rightarrow y \geq -5 \Rightarrow R_f = [-5, \infty)$.

Further, as $-3 \in D_f$, $f(-3)$ exists and $f(-3) = 3 \cdot (-3)^2 - 5 = 22$.

As $43 \in R_f$, on putting $y = 43$ in (i), we get

$3x^2 - 5 = 43 \Rightarrow 3x^2 = 48 \Rightarrow x^2 = 16 \Rightarrow x = -4, 4$.

Therefore, $-4$ and $4$ are numbers (in $D_f$) which are associated with the number 43 in its range.

**Example 8.** *Find the domain of the function $f$ defined by $f(x) = \sqrt{a^2 - x^2}$, $a > 0$.*

**Solution.** Given $f(x) = \sqrt{a^2 - x^2}$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow\ \sqrt{a^2 - x^2}$ must be a real number

$\Rightarrow\ a^2 - x^2 \geq 0 \Rightarrow x^2 - a^2 \leq 0$

$\Rightarrow\ (x + a)(x - a) \leq 0 \Rightarrow -a \leq x \leq a$

$\Rightarrow\ D_f = [-a, a]$.

**Example 9.** *Find the domain and the range of the following functions :*

(i) $f(x) = \sqrt{x - 1}$ (ii) $f(x) = \dfrac{1}{\sqrt{5 - x}}$.

**Solution.** (i) Given $f(x) = \sqrt{x - 1}$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow\ \sqrt{x - 1}$ must be a real number

$\Rightarrow\ x - 1 \geq 0 \Rightarrow x \geq 1$

$\Rightarrow\ D_f = [1, \infty)$.

For $R_f$, let $y = f(x) = \sqrt{x-1}$.

As $x \geq 1$, $x - 1 \geq 0$

$\Rightarrow \sqrt{x-1} \geq 0 \Rightarrow y \geq 0$

$\Rightarrow R_f = [0, \infty)$.

(*ii*) Given $f(x) = \frac{1}{\sqrt{5-x}}$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow \frac{1}{\sqrt{5-x}}$ must be a real number

$\Rightarrow 5 - x > 0 \Rightarrow 5 > x \Rightarrow x < 5$

$\Rightarrow D_f = (-\infty, 5)$.

For $R_f$, let $y = \frac{1}{\sqrt{5-x}}$.

As $x < 5$, $0 < 5 - x$

$\Rightarrow 5 - x > 0 \Rightarrow \sqrt{5-x} > 0$

$\Rightarrow \frac{1}{\sqrt{5-x}} > 0$ $\left(\because \frac{1}{a} > 0 \text{ if and only if } a > 0\right)$

$\Rightarrow y > 0$

$\Rightarrow R_f = (0, \infty)$.

**Example 10.** *Find the domain and the range of the following functions :*

(*i*) $f(x) = \frac{x-3}{2x+1}$ (*ii*) $f(x) = \frac{x^2}{1+x^2}$

(*iii*) $f(x) = \frac{1}{1-x^2}$ (*iv*) $f(x) = \frac{3}{2-x^2}$

**Solution.** (*i*) Given $f(x) = \frac{x-3}{2x+1}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \frac{x-3}{2x+1}$ must be a real number

$\Rightarrow 2x + 1 \neq 0 \Rightarrow x \neq -\frac{1}{2}$

$\Rightarrow D_f$ = set of all real numbers except $-\frac{1}{2}$ *i.e.* $D_f = \mathbf{R} - \left\{-\frac{1}{2}\right\}$.

For $R_f$, let $y = \frac{x-3}{2x+1} \Rightarrow 2xy + y = x - 3$

$\Rightarrow (2y - 1)x = -y - 3 \Rightarrow x = \frac{y+3}{1-2y}$ but $x \in \mathbf{R}$

$\Rightarrow \frac{y+3}{1-2y}$ must be a real number $\Rightarrow 1 - 2y \neq 0 \Rightarrow y \neq \frac{1}{2}$

$\Rightarrow R_f$ = set of all real numbers except $\frac{1}{2}$ *i.e.* $R_f = \mathbf{R} - \left\{\frac{1}{2}\right\}$.

(*ii*) Given $f(x) = \frac{x^2}{1+x^2}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \frac{x^2}{1+x^2}$ must be a real number

$(\because x^2 + 1 \neq 0 \text{ for all } x \in \mathbf{R})$

$\Rightarrow D_f = \mathbf{R}$

For $R_f$, let $y = \frac{x^2}{1+x^2} \Rightarrow x^2y + y = x^2$

$\Rightarrow (y - 1)x^2 = -y \Rightarrow x^2 = -\frac{y}{y-1}$, $y \neq 1$.

But $x^2 \geq 0$ for all $x \in \mathbf{R} \Rightarrow -\frac{y}{y-1} \geq 0,\ y \neq 1$

(Multiply both sides by $(y-1)^2$, a positive real number.)

$\Rightarrow \quad -y(y-1) \geq 0 \Rightarrow y(y-1) \leq 0$

$\Rightarrow \quad (y-0)(y-1) \leq 0 \Rightarrow 0 \leq y \leq 1$ but $y \neq 1$

$\Rightarrow \quad 0 \leq y < 1 \Rightarrow R_f = [0, 1)$.

(*iii*) Given $f(x) = \frac{1}{1-x^2}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \frac{1}{1-x^2}$ must be a real number

$\Rightarrow \quad 1 - x^2 \neq 0 \Rightarrow x \neq -1, 1$

$\Rightarrow \quad D_f$ = set of all real numbers except $-1, 1$ *i.e.* $D_f = \mathbf{R} - \{-1, 1\}$.

For $R_f$, let $y = \frac{1}{1-x^2},\ y \neq 0$

$\Rightarrow \quad 1 - x^2 = \frac{1}{y} \Rightarrow x^2 = 1 - \frac{1}{y},\ y \neq 0.$

But $x^2 \geq 0$ for all $x \in D_f \Rightarrow 1 - \frac{1}{y} \geq 0$ but $y^2 > 0,\ y \neq 0$

(Multiply both sides by $y^2$, a positive real number.)

$\Rightarrow \quad y^2\left(1 - \frac{1}{y}\right) \geq 0 \Rightarrow y(y-1) \geq 0 \Rightarrow (y-0)(y-1) \geq 0$

$\Rightarrow \quad$ either $y \leq 0$ or $y \geq 1$ but $y \neq 0$

$\Rightarrow \quad R_f = (-\infty, 0) \cup [1, \infty)$.

(*iv*) Given $f(x) = \frac{3}{2-x^2}$.

For $D_f$, $2 - x^2 \neq 0 \Rightarrow x \neq \pm\sqrt{2} \Rightarrow D_f = \mathbf{R} - \left\{-\sqrt{2}, \sqrt{2}\right\}$.

For $R_f$, $y = \frac{3}{2-x^2} \Rightarrow 2 - x^2 = \frac{3}{y},\ y \neq 0 \Rightarrow x^2 = 2 - \frac{3}{y},\ y \neq 0.$

As $x^2 \geq 0$ for all $x \in D_f$, $2 - \frac{3}{y} \geq 0,\ y \neq 0$ but $y^2 > 0$

$\Rightarrow \quad y^2\left(2 - \frac{3}{y}\right) \geq 0,\ y \neq 0 \Rightarrow y(2y-3) \geq 0,\ y \neq 0 \Rightarrow y\left(y - \frac{3}{2}\right) \geq 0,\ y \neq 0$

$\Rightarrow \quad y \leq 0$ or $y \geq \frac{3}{2},\ y \neq 0 \Rightarrow R_f = (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$.

**Example 11.** *Find the domain and the range of the following functions :*

(*i*) $f(x) = \frac{x}{1+x^2}$ (*ii*) $f(x) = \frac{x^2-x+1}{x^2+x+1}$.

**Solution.** (*i*) Given $f(x) = \frac{x}{1+x^2}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \frac{x}{1+x^2}$ must be a real number

$\Rightarrow \quad D_f = \mathbf{R}$ ($\because x^2 + 1 \neq 0$ for all $x \in \mathbf{R}$)

For $R_f$, let $y = \frac{x}{1+x^2}$ ...(1)

As $0 \in D_f$, on putting $x = 0$ in (1), we get $y = 0 \Rightarrow 0 \in R_f$.

When $y \neq 0$, from (1), we get $y + yx^2 = x$

$\Rightarrow$ $yx^2 - x + y = 0$, which is a quadratic in $x$ having real roots $(\because x \in \mathbf{R})$

$\Rightarrow$ discriminant $\geq 0$ $\Rightarrow (-1)^2 - 4 . y . y \geq 0$

$\Rightarrow$ $1 - 4y^2 \geq 0$ $\Rightarrow 4y^2 - 1 \leq 0$

$\Rightarrow$ $y^2 - \frac{1}{4} \leq 0 \Rightarrow \left(y + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) \leq 0$

$\Rightarrow$ $-\frac{1}{2} \leq y \leq \frac{1}{2}$, $y \neq 0$ but $0 \in R_f \Rightarrow R_f = \left[-\frac{1}{2}, \frac{1}{2}\right]$.

(*ii*) Given $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \frac{x^2 - x + 1}{x^2 + x + 1}$ must be a real number

$\Rightarrow$ $x^2 + x + 1 \neq 0$.

Now $x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2} \notin \mathbf{R}$

$\Rightarrow$ $x^2 + x + 1 \neq 0$ for all $x \in \mathbf{R}$ $\Rightarrow D_f = \mathbf{R}$.

For $R_f$, let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$ ...(1)

As $0 \in D_f$, on putting $x = 0$ in (1), we get $y = 1 \Rightarrow 1 \in R_f$.

When $y \neq 1$, from (1), we get $(x^2 + x + 1)\, y = x^2 - x + 1$

$\Rightarrow$ $(y - 1)\, x^2 + (y + 1)\, x + (y - 1) = 0$

which is a quadratic in $x$ having real roots $(\because x \in \mathbf{R})$

$\Rightarrow$ discriminant $\geq 0$ $\Rightarrow (y + 1)^2 - 4(y - 1)(y - 1) \geq 0$

$\Rightarrow$ $y^2 + 2y + 1 - 4y^2 + 8y - 4 \geq 0$

$\Rightarrow$ $-3y^2 + 10y - 3 \geq 0$ $\Rightarrow y^2 - \frac{10}{3}y + 1 \leq 0 \Rightarrow \left(y - \frac{1}{3}\right)(y - 3) \leq 0$

$\Rightarrow$ $\frac{1}{3} \leq y \leq 3$, $y \neq 1$ but $1 \in R_f \Rightarrow R_f = \left[\frac{1}{3}, 3\right]$.

**Example 12.** *Find the domain and the range of the following functions :*

(*i*) $f(x) = \sqrt{x^2 - 4}$ (*ii*) $f(x) = \sqrt{9 - x^2}$ (*iii*) $f(x) = \frac{1}{\sqrt{9 - x^2}}$.

**Solution.** (*i*) Given $f(x) = \sqrt{x^2 - 4}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \sqrt{x^2 - 4}$ must be a real number

$\Rightarrow$ $x^2 - 4 \geq 0$ $\Rightarrow (x + 2)(x - 2) \geq 0$

$\Rightarrow$ either $x \leq -2$ or $x \geq 2$

$\Rightarrow$ $D_f = (-\infty, -2] \cup [2, \infty)$.

For $R_f$, let $y = \sqrt{x^2 - 4}$ ...(1)

As square root of a real number is always non-negative, $y \geq 0$.

On squaring (1), we get $y^2 = x^2 - 4$

$\Rightarrow$ $x^2 = y^2 + 4$ but $x^2 \geq 0$ for all $x \in D_f$

$\Rightarrow$ $y^2 + 4 \geq 0$ $\Rightarrow y^2 \geq -4$, which is true for all $y \in \mathbf{R}$. Also $y \geq 0$

$\Rightarrow$ $R_f = [0, \infty)$.

(ii) Given $f(x) = \sqrt{9-x^2}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \sqrt{9-x^2}$ must be a real number

$\Rightarrow \quad 9 - x^2 \geq 0 \Rightarrow -(x^2 - 9) \geq 0 \Rightarrow x^2 - 9 \leq 0$

$\Rightarrow \quad (x+3)(x-3) \leq 0 \Rightarrow -3 \leq x \leq 3 \Rightarrow D_f = [-3, 3]$.

For $R_f$, let $y = \sqrt{9-x^2}$ ...(1)

As square root of a real number is always non-negative, $y \geq 0$.

Squaring (1), we get

$y^2 = 9 - x^2$

$\Rightarrow \quad x^2 = 9 - y^2$ but $x^2 \geq 0$ for all $x \in D_f$

$\Rightarrow \quad 9 - y^2 \geq 0 \Rightarrow -(y^2 - 9) \geq 0 \Rightarrow y^2 - 9 \leq 0$

$\Rightarrow \quad (y+3)(y-3) \leq 0 \Rightarrow -3 \leq y \leq 3$ but $y \geq 0$

$\Rightarrow \quad R_f = [0, 3]$.

(iii) Given $f(x) = \dfrac{1}{\sqrt{9-x^2}}$.

For $D_f$, $f(x)$ must be a real number $\Rightarrow \dfrac{1}{\sqrt{9-x^2}}$ must be a real number

$\Rightarrow \quad 9 - x^2 > 0 \Rightarrow -(x^2 - 9) > 0 \Rightarrow x^2 - 9 < 0$

$\Rightarrow \quad (x+3)(x-3) < 0 \Rightarrow -3 < x < 3 \Rightarrow D_f = (-3, 3)$.

For $R_f$, let $y = \dfrac{1}{\sqrt{9-x^2}}$, $y \neq 0$ ...(1)

Also as the square root of a real number is always non-negative, $y > 0$.

On squaring (1), we get

$y^2 = \dfrac{1}{9-x^2} \Rightarrow 9 - x^2 = \dfrac{1}{y^2} \Rightarrow x^2 = 9 - \dfrac{1}{y^2}$.

But $x^2 \geq 0$ for all $x \in D_f \Rightarrow 9 - \dfrac{1}{y^2} \geq 0$ but $y^2 > 0$

(Multiply both sides by $y^2$, a positive real number)

$\Rightarrow \quad 9y^2 - 1 \geq 0 \Rightarrow y^2 - \dfrac{1}{9} \geq 0 \Rightarrow \left(y + \dfrac{1}{3}\right)\left(y - \dfrac{1}{3}\right) \geq 0$

$\Rightarrow$ either $y \leq -\dfrac{1}{3}$ or $y \geq \dfrac{1}{3}$ but $y > 0 \Rightarrow y \geq \dfrac{1}{3}$

$\Rightarrow \quad R_f = \left[\dfrac{1}{3}, \infty\right)$.

**Example 13.** *Find the domain and the range of the following functions :*

(i) $f(x) = -|x|$ (ii) $f(x) = 1 - |x-2|$.

**Solution.** (i) Given $f(x) = -|x|$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow -|x|$ must be a real number, which is a real number for every $x \in \mathbf{R}$

$\Rightarrow \quad D_f = \mathbf{R}$.

For $R_f$, let $y = f(x) = -|x|$.

Since $|x| \geq 0$ for all $x \in \mathbf{R} \Rightarrow -|x| \leq 0$ for all $x \in \mathbf{R}$

$\Rightarrow \quad y \leq 0 \Rightarrow R_f = (-\infty, 0]$.

(ii) Given $f(x) = 1 - |x - 2|$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow 1 - |x - 2|$ must be a real number, which is a real number for every $x \in \mathbf{R}$

$\Rightarrow \quad D_f = \mathbf{R}$.

For $R_f$, let $y = f(x) = 1 - |x - 2|$.

Since $|x| \geq 0$ for all $x \in \mathbf{R} \Rightarrow |x - 2| \geq 0$ for all $x \in \mathbf{R}$

$\Rightarrow \quad -|x - 2| \leq 0 \Rightarrow 1 - |x - 2| \leq 1 \Rightarrow y \leq 1$

$\Rightarrow \quad R_f = (-\infty, 1]$.

**Example 14.** *Find the domain and the range of the function f defined by* $f(x) = \frac{x+2}{|x+2|}$.

**Solution.** Given $f(x) = \frac{x+2}{|x+2|}$

For $D_f$, $f(x)$ must be a real number $\Rightarrow \frac{x+2}{|x+2|}$ must be a real number

$\Rightarrow \quad |x + 2| \neq 0 \Rightarrow x + 2 \neq 0 \Rightarrow x \neq -2$

$\Rightarrow \quad D_f$ = set of all real numbers except $-2$ *i.e.* $D_f = \mathbf{R} - \{-2\}$

For $R_f$, let $y = \frac{x+2}{|x+2|}$. Two cases arise.

**Case I.** If $x + 2 > 0$, then $|x + 2| = x + 2$,

$\therefore \quad y = \frac{x+2}{|x+2|} = 1$.

**Case II.** If $x + 2 < 0$, $|x + 2| = -(x + 2)$,

$\therefore \quad y = \frac{x+2}{|x+2|} = \frac{x+2}{-(x+2)} = -1$.

$\therefore$ On combining two cases, $R_f = \{-1, 1\}$.

**Example 15.** *Find the domain of the following functions :*

(i) $f(x) = \frac{1}{\sqrt{x + |x|}}$ (ii) $f(x) = \frac{1}{\sqrt{x - |x|}}$.

**Solution.** (i) Given $f(x) = \frac{1}{\sqrt{x + |x|}}$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow \quad \frac{1}{\sqrt{x + |x|}}$ must be a real number

$\Rightarrow \quad x + |x| > 0 \Rightarrow x > 0$

($\because |x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0 \Rightarrow x + |x| = 0$ if $x \leq 0$)

$\Rightarrow \quad D_f = (0, \infty)$.

(ii) Given $f(x) = \frac{1}{\sqrt{x - |x|}}$.

For $D_f$, $f(x)$ must be a real number

$\Rightarrow \quad \frac{1}{\sqrt{x - |x|}}$ must be a real number

$\Rightarrow \quad x - |x| > 0 \Rightarrow x > |x|$

$\Rightarrow \quad |x| < x$ but $|x| \geq x$ for all $x \in \mathbf{R}$

$\Rightarrow \quad D_f = \phi$, so the given function is not defined.

**Example 16.** *Find the domain for which the functions $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.*

**Solution.** As the functions $f$ and $g$ are equal, $f(x) = g(x)$

$\Rightarrow \quad 2x^2 - 1 = 1 - 3x \Rightarrow 2x^2 + 3x - 2 = 0$

$\Rightarrow \quad 2x^2 + 4x - x - 2 = 0 \Rightarrow 2x(x + 2) - 1(x + 2) = 0$

$\Rightarrow \quad (x + 2)(2x - 1) = 0 \Rightarrow x + 2 = 0$ or $2x - 1 = 0$

$\Rightarrow \quad x = -2$ or $x = \frac{1}{2}$.

Hence, the domain for which the functions $f$ and $g$ are equal is $\left\{-2, \frac{1}{2}\right\}$.

## EXERCISE 2.4

*Very short answer type questions (1 to 15) :*

1. Let N be the set of natural numbers. Define a real valued function $f : \mathbf{N} \to \mathbf{N}$ by $f(x) = 2x - 1$. Using this definition complete the table given below :

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | ... | ... | ... | ... | ... | ... | ... |

2. If $f(x) = x^2$, find $\frac{f(1.1) - f(1)}{1.1 - 1}$.

3. A function is defined by $f(x) = \frac{3x^2 + 2x - 1}{x + 1}$, $x \in \mathbf{R}$, $x \neq -1$. Find the value of $f(-3) + 1$.

4. Is the relation $f$ defined by $f(x) = \begin{cases} x^2, & 0 \le x \le 3 \\ 2x, & 3 < x \le 5 \end{cases}$ a function ? Justify your answer.

5. A real function $f$ is defined by $f(x) = 3x - 2$, $x \in \mathbf{R}$, find the value of $f(-3) + f(0) \times f(7)$.

6. If a real function $f$ is defined by $f(x) = 2x^2 - 3$, find $x$ such $f(x) = 15$.

7. Given $f(x) = x^3 - 1$, find $x$ if $f(x)$ is 215.

8. A function is given by the formula $f(x) = 144 - 16x^2$. Calculate $f(2)$. Also find the values of $x$ when $f(x) = 0$.

9. A function '$f$' is defined by $f(x) = 2x^2 + 3$, for all $x \in \mathbf{R}$. Find
   (*i*) image of $-1$ under $f$
   (*ii*) element (elements) of the domain which has image 35.

10. If $f(x) = 2x^2 + 1$ and domain of $f = \{-2, -1, 0, 1, 3\}$, find
   (*i*) range of $f$ (*ii*) $f(3) \times f(-2)$ (*iii*) $x$ if $f(x) = 9$.

11. A function '$f$' is defined by $f(x) = x^2 + 3$, $x \in \mathbf{N}$ and $x \le 5$.
   (*i*) Find the range of $f$. (*ii*) Find $f(2) \times f(5)$.
   (*iii*) Does $f(-3)$ exist ? (*iv*) Find $x$ when $f(x) = 7$.

12. Find the domain of the following real functions :
   (*i*) $f(x) = \frac{x^2 - 4}{x + 2}$ (*ii*) $f(x) = \frac{3x}{28 - x}$
   (*iii*) $f(x) = \frac{x}{x^2 + 3x + 2}$ (*iv*) $f(x) = \frac{x^3 - x + 3}{x^2 - 1}$.

13. Find the domain of the following real functions:
   (*i*) $f(x) = \sqrt{x - 2}$ (*ii*) $f(x) = \sqrt{2 - 3x}$ (*iii*) $f(x) = \frac{1}{\sqrt{x - 1}}$ (*iv*) $f(x) = \frac{1}{\sqrt{3 - x}}$.

14. Find the domain of the following real functions :

(i) $f(x) = |x + 2|$ (ii) $f(x) = -|x - 2|$ (iii) $f(x) = 2 - |x - 1|$ (iv) $f(x) = \frac{|x| - x}{2x}$.

15. Find the range of the following functions :

(i) $f(x) = x^2 + 2, x \in \mathbb{R}$ (ii) $f(x) = 3 - 2x, x \in \mathbb{R}, x \geq 1$.

16. Find the domain and the range of the function $f(x) = 2x^2 + 1$. Also find $f(-2)$ and the numbers which are associated with the number 51 in its range.

17. Find the domain and the range of the following functions :

(i) $f(x) = \sqrt{x + 2}$ (ii) $f(x) = \sqrt{3 - 2x}$ (iii) $f(x) = \frac{1}{\sqrt{x - 5}}$.

18. Find the domain and the range of the following functions :

(i) $f(x) = \frac{4 - x}{x - 4}$ (ii) $f(x) = \frac{x^2 - 9}{x - 3}$

(iii) $f(x) = \frac{x + 1}{2x + 1}$ (iv) $f(x) = \frac{4 + x}{4 - x}$

19. Find the domain and the range of the following functions :

(i) $f(x) = \sqrt{16 - x^2}$ (ii) $f(x) = \sqrt{x^2 - 9}$ (iii) $f(x) = \frac{1}{\sqrt{4 - x^2}}$.

20. Find the domain and the range of the following functions :

(i) $f(x) = |x - 3|$ (ii) $f(x) = 3 - |x - 2|$.

21. If a real function $f$ is defined by $f(x) = \frac{|x| - x}{2x}$, then find its domain and range.

22. Find the domain and the range of the function $f$ defined by $f(x) = \frac{|x - 4|}{x - 4}$.

23. If $f$ and $g$ are two real functions defined by $f(x) = 2x + 1$ and $g(x) = 4x - 7$, then for what real numbers $x$

(i) $f(x) = g(x)$? (ii) $f(x) < g(x)$?

24. Find the domain for which the functions $f(x) = 3x^2 - 1$ and $g(x) = 3 + x$ are equal.

## 2.6 GRAPH OF A REAL FUNCTION

Graphs provide us geometric visualisation of the problems, and hence; these help a lot in understanding the problems.

**Graph of a real function** '$f$', *denoted by* $G(f)$, *is the set of all points* $(x, f(x))$ *of the plane where* $x \in D_f$. *Precisely,*

$$G(f) = \{(x, f(x)) ; \text{ for all } x \in D_f\}.$$

If the domain of a function '$f$' consists of finitely many points, then the graph of $f$ can be easily plotted in a plane; but if the domain of $f$ consists of infinitely many points, it may not be possible to plot all these points. In such cases, we plot enough points to get an idea of the shape of the graph and then join these points by a free-hand drawing.

The following properties of a graph are worth observation :

(i) *The graph of a function $f$ is a subset of the plane and it is uniquely determined by the function.*

(ii) *For each $c \in D_f$, there exists exactly one point $(c, f(c))$ in $G(f)$, for the value of $f$ at $c$ is uniquely determined.*

Geometrically, it means that if $c \in D_f$, then the vertical line $x = c$ will meet the graph in one point only. On the other hand, if a real number $c$ does not belong to $D_f$, the line $x = c$ shall not meet $G(f)$ at all.

It follows that every subset of a plane cannot form a graph of some function. In order that a (non-empty) set G of points in the plane may form a graph of some function, it is necessary that each vertical line should meet it in atmost one point. The converse is also true *i.e.* any set of points in a plane which does not satisfy this property cannot be the graph of a real valued function.

***For example :***

(*i*) the following graphs represent functions :

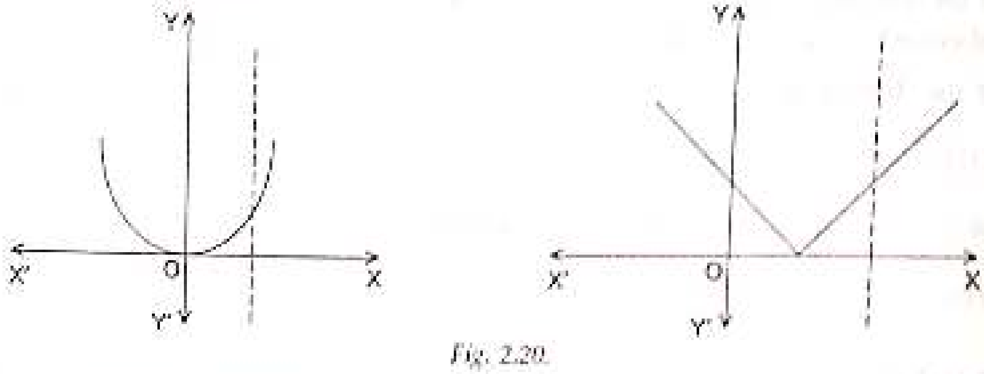



*Fig. 2.20.*

(*ii*) the following graphs do not represent functions :

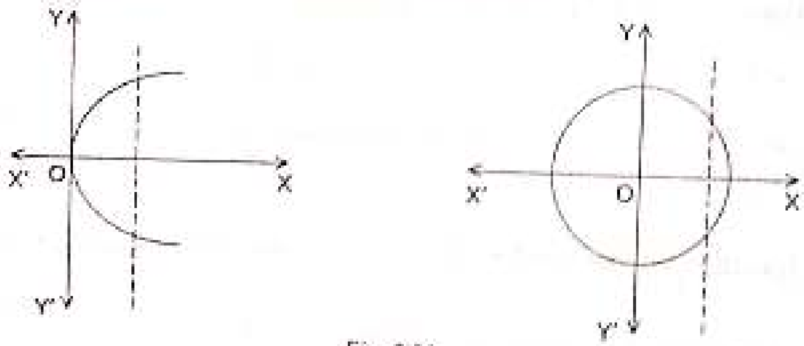



*Fig. 2.21.*

*The set of all x-coordinates of the points of G will constitute the domain of f and their corresponding y-coordinates will form the range of f.*

**REMARK**

We can find the range of a function from its graph. Infact, the values of $y$ constitute the range of the function.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Draw the graphs of the following real functions and hence find their range :*

(*i*) $f(x) = 2x - 1$ (*ii*) $f(x) = \frac{x^2 - 1}{x - 1}$.

**Solution.** (*i*) Given $f(x) = 2x - 1$

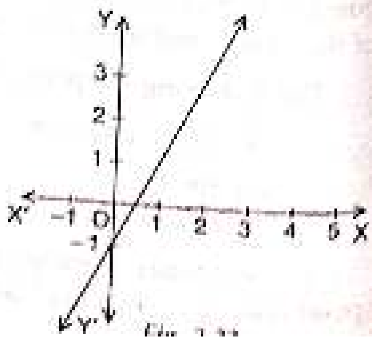
$\Rightarrow \quad D_f = \mathbf{R}$.

Let $y = f(x)$ *i.e.* $y = 2x - 1$, which is a first degree equation in $x$, $y$ and hence it represents a straight line. Two points are sufficient to determine a straight line uniquely.

Table of values

| | | |
|---|---|---|
| $x$ | 0 | 1 |
| $y$ | –1 | 1 |

A portion of the graph is shown in fig. 2.22. From the graph, it is clear that $y$ takes all real values. It follows that $R_f = \mathbf{R}$.



(ii) Given $f(x) = \frac{x^2-1}{x-1} \Rightarrow D_f = \mathbf{R} - \{1\}$

Let $y = f(x) = \frac{x^2-1}{x-1} = x + 1$ $(\because x \neq 1)$

*i.e.* $y = x + 1$, which is a first degree equation in $x$, $y$ and hence it represents a straight line. Two points are sufficient to determine a straight line uniquely.

Table of values

| $x$ | $-1$ | $0$ |
|---|---|---|
| $y$ | $0$ | $1$ |

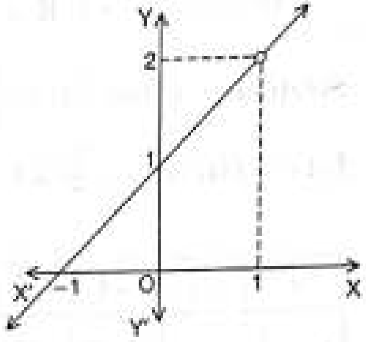


*Fig. 2.23.*

A portion of the graph is shown in fig. 2.23.

From the graph, it is clear that $y$ takes all real values except 2. It follows that $R_f = \mathbf{R} - \{2\}$.

**Example 2.** *Draw the graphs of the following real functions and hence find their range :*

(i) $f(x) = x^2$ (ii) $f(x) = x^3$.

**Solution.** (i) Given $f(x) = x^2 \Rightarrow D_f = \mathbf{R}$.

Let $y = f(x) = x^2$, $x \in \mathbf{R}$

Table of values

| $x$ | ... | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = x^2$ | ... | 16 | 9 | 4 | 1 | 0 | 1 | 4 | 9 | 16 | ... |

Plot the points $(-4, 16)$, $(-3, 9)$, $(-2, 4)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$, $(4, 16)$, ..., and join these points by a free hand drawing.

A portion of the graph is shown in fig. 2.24.

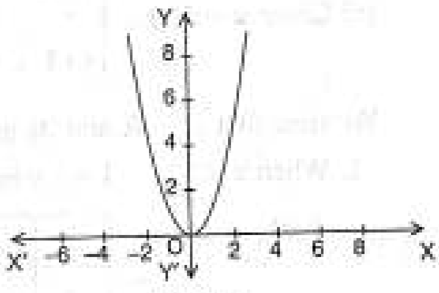


*Fig. 2.24.*

From the graph, it is clear that $y$ takes all non-negative real values. It follows that $R_f = [0, \infty)$.

(ii) Given $f(x) = x^3 \Rightarrow D_f = \mathbf{R}$.

Let $y = f(x) = x^3$, $x \in \mathbf{R}$.

Table of values

| $x$ | ... | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | ... |
|---|---|---|---|---|---|---|---|---|---|
| $y = x^3$ | ... | $-27$ | $-8$ | $-1$ | $0$ | $1$ | $8$ | $27$ | ... |

Plot the points $(-3, -27)$, $(-2, -8)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, $(2, 8)$, $(3, 27)$, ..., and join these points by a free hand drawing.

A portion of the graph is shown in fig. 2.25.

From the graph, it is clear that $y$ takes all real values.

Hence, $R_f = \mathbf{R}$.

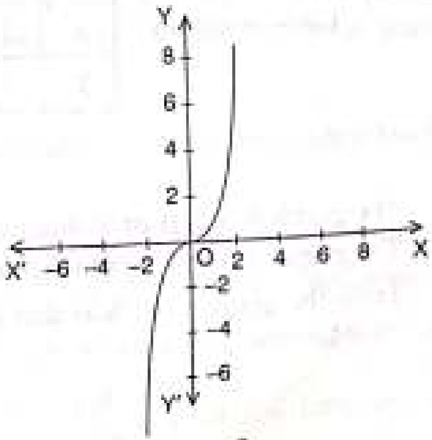


*Fig. 2.25.*

**Example 3.** *Draw the graphs of the following real functions and hence find their range :*

(i) $f(x) = \frac{1}{x}, x \in R, x \neq 0$

(ii) $f(x) = \begin{cases} 1 - x &, x < 0 \\ 1 &, x = 0 \\ x + 1 &, x > 0. \end{cases}$

**Solution.** (i) Given $f(x) = \frac{1}{x}, x \in R, x \neq 0$.

Let $y = f(x)$ i.e. $y = \frac{1}{x}, x \in R, x \neq 0$.

Table of values

| $x$ | ... | $-4$ | $-2$ | $-1$ | $-0.5$ | $-0.25$ | 0.25 | 0.5 | 1 | 2 | 4 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = \frac{1}{x}$ | ... | $-0.25$ | $-0.5$ | $-1$ | $-2$ | $-4$ | 4 | 2 | 1 | 0.5 | 0.25 | ... |

Plot the points shown in the above table and join these points by a free hand drawing.

Portions of the graph are shown in fig. 2.26.

From the graph, it is clear that

$R_f = R - \{0\}$.

This function is called **reciprocal function.**

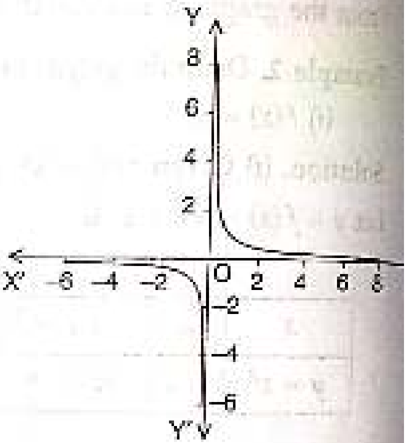



*Fig. 2.26.*

(ii) Given $y = f(x) = \begin{cases} 1 - x &, x < 0 \\ 1 &, x = 0 \\ x + 1 &, x > 0. \end{cases}$

We note that $D_f = R$ and its graph consists of the following three parts :

1. When $x < 0$, $y = 1 - x$ which is a first degree equation in $x, y$.

Table of values

| $x$ | $-1$ | $-2$ |
|---|---|---|
| $y$ | 2 | 3 |

The graph is a part of a straight line.

2. When $x = 0$, $y = 1$. It represents a point (0, 1).

3. When $x > 0$, $y = 1 + x$ which is a first degree equation in $x, y$.

Table of values

| $x$ | 1 | 2 |
|---|---|---|
| $y$ | 2 | 3 |

The graph is a part of a straight line.

The graph of the given function is shown in fig. 2.27.

From the graph, it is clear that $R_f = [1, \infty)$.

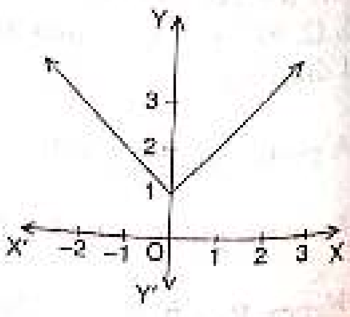



*Fig. 2.27.*

## EXERCISE 2.5

*Draw the graphs of the following (1 to 4) real functions. Hence, find their range.*

1. (*i*) $f(x) = x + 10$ (*ii*) $f(x) = 1 - 2x$.

2. (*i*) $f(x) = \frac{x^2 - 4}{x - 2}$ (*ii*) $f(x) = \begin{cases} x + 3 & , x < 2 \\ 2x + 1 & , x \geq 2. \end{cases}$

3. (*i*) $f(x) = \begin{cases} x - 1 & , x < 2 \\ 2x + 3 & , x \geq 2 \end{cases}$ (*ii*) $f(x) = \begin{cases} -1 & , x \leq -1 \\ x & , -1 < x < 1 \\ 1 & , x \geq 1. \end{cases}$

4. (*i*) $f(x) = \frac{1}{2}x^2$ (*ii*) $f(x) = 1 - x^2$.

## 2.7 SOME REAL FUNCTIONS

Now we shall discuss some standard real functions which are frequently used in calculus.

### 1. Constant function

Let $c$ be a fixed real number, then a function $f$ defined by $f(x) = c$ for all $x \in \mathbf{R}$ is called a ***constant function***.

Its graph is a straight line parallel to $x$-axis at a distance $|c|$ units from it. A portion of the graph for $c > 0$ is shown in fig. 2.28. Clearly, $D_f = \mathbf{R}$ and $R_f = \{c\}$.

In particular, if $c = 0$, then $f$ is called ***zero function*** and its graph is the $x$-axis itself.

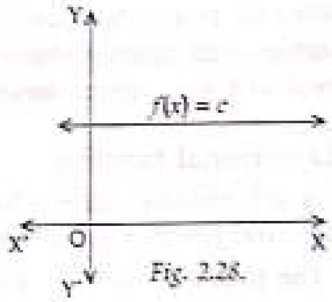


*Fig. 2.28.*

### 2. Identity function

The function $f$ defined by $f(x) = x$ for all $x \in \mathbf{R}$ is called the ***identity function***. Its graph is a straight line passing through origin and inclined at an angle of 45° with the $x$-axis. A portion of the graph is shown in fig. 2.29. Clearly, $D_f = \mathbf{R}$ and $R_f = \mathbf{R}$.

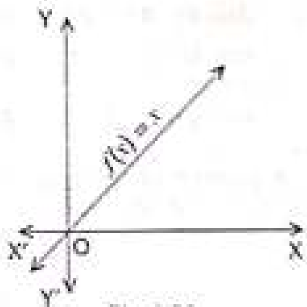


*Fig. 2.29.*

### 3. Polynomial function

A function $f$ defined by $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ where $a_0, a_1, a_2, \ldots, a_n$ are all real numbers and $n$ is a non-negative integer is called a ***polynomial function***; $D_f = \mathbf{R}$.

If $a_n \neq 0$, then $f$ is called a polynomial function of ***degree n***. If $n = 1$, it is called a ***linear function*** and if $n = 2$, it is called a ***quadratic function*** etc. For example, $f(x) = 7 - 9x$ is a *linear function* and $f(x) = 5 + \sqrt{3}x - 7x^2$ is a *quadratic function*.

In particular, if $a_i = 0$ for all $i \geq 1$, then the polynomial function reduces to the *constant function* $f(x) = a_0$.

### 4. Rational function

A function which can be expressed as the quotient of two polynomial functions *i.e.* $f(x) = \frac{g(x)}{h(x)}$ where $g(x)$, $h(x)$ are polynomial functions and $h(x) \neq 0$ (zero polynomial) is called a ***rational function***. For example, the function $f(x) = \frac{3 + 7x^2 - 8x^3}{4 - x^2}$ is a rational function with $D_f = \mathbf{R} - \{-2, 2\}$.

**5. Modulus function.**

The function $f$ defined by

$$f(x) = |x| \text{ i.e. } y = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

is called the *modulus* (or *absolute value*) *function*.

$D_f = R$, a portion of its graph is shown in fig. 2.30.

Clearly, $R_f = [0, \infty)$.

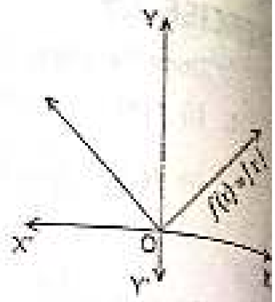



Fig. 2.30.

**6. Signum function**

The function $f$ defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \quad \text{i.e. } y = \begin{cases} 1 & , x > 0 \\ 0 & , x = 0 \\ -1 & , x < 0 \end{cases}$$

is called the *signum function*. Its $D_f = R$ and $R_f = \{-1, 0, 1\}$. A portion of its graph is shown in fig. 2.31. Note that $f$ has a break at $x = 0$. Signum function is denoted by sgn $x$.

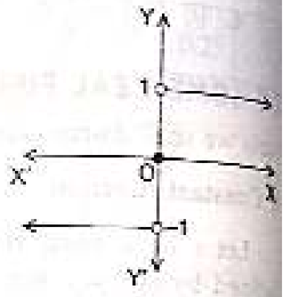



Fig. 2.31.

**7. Exponential function**

If $a$ is any positive real number, then the function $f$ defined by $f(x) = a^x$ is called the *general exponential function*. $D_f$ (domain of $f$) = R.

**The function $f(x) = a^x$ ($a > 0$, $x \in R$) has the following properties :**

(*i*) $a^0 = 1$.

(*ii*) $a^x \cdot a^y = a^{x+y}$ for all $x, y \in R$.

(*iii*) $(a^x)^y = a^{xy}$ for all $x, y \in R$.

(*iv*) $a^{-x} = \dfrac{1}{a^x}$ for all $x \in R$.

A portion of the graph is shown in fig. 2.32.

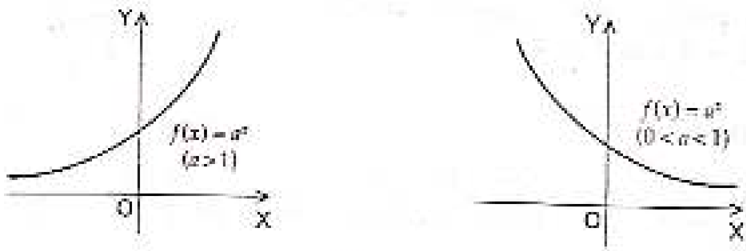



Fig. 2.32.

In particular, if $a = 1$ then $f(x) = 1^x = 1$, which is a constant function.

Note that if $a \neq 1$, then $R_f$ (range of $f$) = $(0, \infty)$ and if $a = 1$, then $R_f = \{1\}$.

In particular, if $a = e$ i.e. the base is $e$ (an irrational number whose approximate value is 2.71828), then the function $f(x) = e^x$ is called **(*natural*) *exponential function*.**

$D_f = R$ and $R_f = (0, \infty)$.

**8. Logarithmic function**

If $a$ is a positive real number, $a \neq 1$, and $x$ is a positive real number then the *logarithmic function* with base $a$ is denoted by the symbol $\log_a x$ and is defined as :

$$y = \log_a x \text{ if and only if } x = a^y.$$

The domain of logarithmic function is $(0, \infty)$ and its range is R.

A portion of the graph is shown in fig. 2.33.

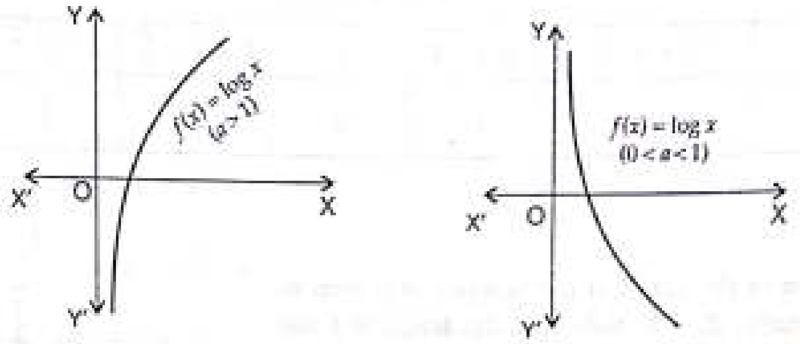



*Fig. 2.33.*

**REMARK**

When $x$ is any non-zero real number, $|x|$ will be a positive real number, so $\log_a |x|$ $(a > 0, \neq 1)$ makes sense for all $x$ except zero.

In particular, when the base is $e$, the function $f(x) = \log_e x$ is called **natural logarithmic function**. Sometimes, it is denoted by $\log x$ or $\ln x$. Its domain is $(0, \infty)$ and $R_f = R$.

**NOTE**

When the base of a logarithmic function is not mentioned, it is always to be taken $e$.

**The function $\log_a x$ $(a > 0, \neq 1)$ has the following properties :**

(*i*) Domain is $(0, \infty)$ and range is R.

(*ii*) $\log_a 1 = 0$ and $\log_a a = 1$.

(*iii*) $\log_a xy = \log_a x + \log_a y$, $x > 0$, $y > 0$. (Product law)

(*iv*) $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$, $x > 0$, $y > 0$. (Quotient law)

(*v*) $\log_a x^y = y \log_a x$, $x > 0$. (Power law)

(*vi*) $\log_a x = \frac{\log x}{\log a}$, $x > 0$. (Base changing formula)

(*vii*) $a^x = e^{x \log a}$.

(*viii*) $x = e^{\log x}$, $x > 0$.

(*ix*) $a^x = e^{x \log a}$.

(*x*) $x = e^{\log x}$, $x > 0$.

**9. Greatest integer function**

For every $x \in R$, $[x]$ denotes the *greatest integer* less than or equal to $x$ *i.e.* if $x$ is an integer, then $[x] = x$ and if $x$ is not an integer, then $[x]$ is equal to the integer immediately to the left of $x$.

For example, $[5] = 5$, $[-5] = -5$, $[0] = 0$, $\left[\frac{14}{3}\right] = 4$, $[\sqrt{2}] = 1$, $[-\sqrt{3}] = -2$, $\left[-\frac{9}{2}\right] = -5$, $[-\pi] = -4$ etc.

*Some facts about* $[x]$

(*i*) $[x] = x$ iff $x \in$ I (set of integers).

(*ii*) $[x] < x$ iff $x \notin$ I.

(*iii*) $[x] = k$ $(k \in$ I$)$ iff $k \leq x < k + 1$.

(*iv*) If $k \in$ I, then $[x + k] = [x] + k$ for all $x \in$ R.

The function $f$ defined by $f(x) = [x]$ is called the ***greatest integer*** or ***integral part function***. $D_f = R$. It is also called a *step function*.

Table of values

| $x$ | ... | $-2 \le x < -1$ | $-1 \le x < 0$ | $0 \le x < 1$ | $1 \le x < 2$ | $2 \le x < 3$ | ... |
|---|---|---|---|---|---|---|---|
| $y$ | ... | $-2$ | $-1$ | 0 | 1 | 2 | ... |

A portion of the graph of the function is shown in fig. 2.34. Clearly, $R_f = I$. Note that the function $f$ has breaks at all integral values of $x$.

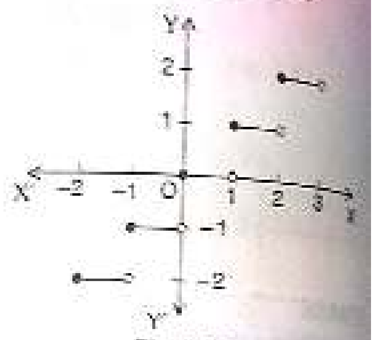


Fig. 2.34.

**10. Fractional part function**

*The function $f$ defined by $f(x) = x - [x]$ is called the **fractional part function**.*

Since the greatest integer function $[x]$ is defined for all $x \in R$, $D_f = R$.

We know that $[x] = k$ $(k \in I)$ if $k \le x < k + 1$.

Put $k = \ldots, -2, -1, 0, 1, 2, \ldots$ and construct the table as under

| $x$ | ... | $-2 \le x < -1$ | $-1 \le x < 0$ | $0 \le x < 1$ | $1 \le x < 2$ | $2 \le x < 3$ | ... |
|---|---|---|---|---|---|---|---|
| $y$ | ... | $x + 2$ | $x + 1$ | $x$ | $x - 1$ | $x - 2$ | ... |

$0 \le x < 1$, $y = x$

| $x$ | 0 | .5 |
|---|---|---|
| $y$ | 0 | .5 |

$1 \le x < 2$, $y = x - 1$

| $x$ | 1 | 1.5 |
|---|---|---|
| $y$ | 0 | .5 |

$2 \le x < 3$, $y = x - 2$

| $x$ | 2 | 2.5 |
|---|---|---|
| $y$ | 0 | .5 |

A portion of the graph is shown in fig. 2.35. Clearly, $R_f = [0, 1)$.

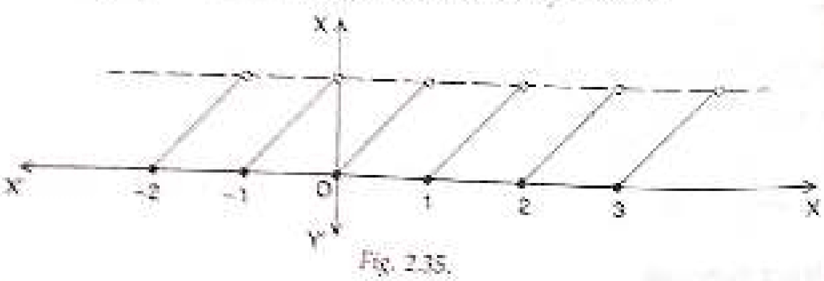


Fig. 2.35.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Solve the following :*

(*i*) $[3x - 4] = 5$ (*ii*) $[1 - 2x] = -3$ (*iii*) $[x]^2 - 5[x] + 6 = 0$.

**Solution.** (*i*) Given $[3x - 4] = 5 \Rightarrow 5 \le 3x - 4 < 6$

($\because$ if $k \in I$, then $[x] = k \Leftrightarrow k \le x < k + 1$)

$\Rightarrow 9 \le 3x < 10 \Rightarrow 3 \le x < \frac{10}{3}$.

Hence, the solution set is $\left[3, \frac{10}{3}\right)$.

(*ii*) Given $[1 - 2x] = -3 \Rightarrow -3 \le 1 - 2x < -2$

$\Rightarrow \quad -4 \le -2x < -3 \Rightarrow 2 \ge x > \frac{3}{2} \Rightarrow \frac{3}{2} < x \le 2.$

Hence, the solution set is $\left(\frac{3}{2}, 2\right]$.

(*iii*) Given $[x]^2 - 5[x] + 6 = 0 \Rightarrow ([x] - 2)([x] - 3) = 0$

$\Rightarrow \quad [x] = 2$ or $[x] = 3$

$\Rightarrow \quad 2 \le x < 3$ or $3 \le x < 4$

$\Rightarrow \quad x \in [2, 4).$

Hence, the solution set is $[2, 4)$.

**Example 2.** *If $f : R \to R$ is defined by $f(x) = 2^x$, then find*

(*i*) *range of f* (*ii*) *x such that $f(x) = 1$.*

*Also, prove that $f(x + y) = f(x) f(y)$.*

**Solution.** (*i*) Given $f : R \to R$ defined by $f(x) = 2^x$.

Note that $D_f = R$ and for all $x \in R$, $2^x > 0$

$\Rightarrow$ range of $f = (0, \infty)$.

(*ii*) Given $f(x) = 1 \Rightarrow 2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0.$

Further, $f(x + y) = 2^{x+y} = 2^x \cdot 2^y = f(x) f(y)$.

**Example 3.** *If $f : R^+ \to R$ is defined by $f(x) = \log_e x$, where $R^+$ is the set of positive real numbers, then find*

(*i*) *range of f* (*ii*) *x such that $f(x) = -1$.*

*Also, prove that $f(xy) = f(x) + f(y)$, for all $x, y \in R^+$.*

**Solution.** (*i*) Given $f(x) = \log_e x$.

We know that $D_f = R^+$ and for all $x \in R^+$, $\log_e x \in R$

$\Rightarrow$ range of $f = R$.

(*ii*) Given $f(x) = -1 \Rightarrow \log_e x = -1$

$\Rightarrow \quad x = e^{-1} \Rightarrow x = \frac{1}{e}.$

Further, $f(xy) = \log_e (xy) = \log_e x + \log_e y$

$\Rightarrow \quad f(xy) = f(x) + f(y).$

**Example 4.** *If $f(x) = \log\left(\frac{1+x}{1-x}\right)$, prove that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.*

**Solution.** Given $f(x) = \log\left(\frac{1+x}{1-x}\right)$.

$$\therefore \quad f\left(\frac{2x}{1+x^2}\right) = \log\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right) = \log\left(\frac{1+x^2+2x}{1+x^2-2x}\right)$$

$$= \log\frac{(1+x)^2}{(1-x)^2} = \log\left(\frac{1+x}{1-x}\right)^2$$

$$= 2\log\left(\frac{1+x}{1-x}\right) = 2f(x).$$

**Example 5.** *Find the domain of the function* $f(x) = \dfrac{1}{\log(3-x)}$.

**Solution.** For $D_f$, $f(x)$ must be a real number

$\Rightarrow \quad \dfrac{1}{\log(3-x)}$ must be a real number

$\Rightarrow \quad 3 - x > 0$ and $3 - x \neq 1$ ($\because \log 1$

$\Rightarrow \quad 3 > x$ and $2 \neq x \Rightarrow x < 3$ and $x \neq 2$

$\Rightarrow \quad D_f = (-\infty, 2) \cup (2, 3)$.

## EXERCISE 2.6

1. Solve the following equations for $x$ :
   (*i*) $[2x - 3] = 5$ (*ii*) $[2x + 1] = -2$ (*iii*) $[2 - 3x] = 5$.
2. If $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$, then show that $f(x) + f(y) = f\left(\dfrac{x+y}{1+xy}\right)$.
3. Find the domain of the following functions :
   (*i*) $\log(4x - 3)$ (*ii*) $\dfrac{1}{\log(9-x^2)}$.

## 2.8 OPERATIONS ON REAL FUNCTIONS

The algebraic operations of addition, subtraction, multiplication and division etc. are performed on two real valued functions *suitably* in the same manner as they are performed two real numbers.

Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be any two real functions where $X \subset \mathbb{R}$, then

(*i*) *the sum of f and g denoted by f + g, is the function defined by*
$(f + g)(x) = f(x) + g(x)$, *for all* $x \in X$.

(*ii*) *the difference of f and g denoted by f − g, is the function defined by*
$(f - g)(x) = f(x) - g(x)$, *for all* $x \in X$.

(*iii*) *the product of f and g denoted by f g, is the function defined by*
$(f g)(x) = f(x)\, g(x)$, *for all* $x \in X$.

(*iv*) *the quotient of f by g denoted by* $\dfrac{f}{g}$, *is the function defined by*
$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, *for all* $x \in X_1$ *where* $X_1 = \{x : x \in X, g(x) \neq 0\}$.

Further, the function $ff$ is usually denoted by $f^2$, the function $f^2 f$ or $ff^2$ by $f^3$ and so on and hence the **power function** $f^n$ $(n \in \mathbb{N})$ is defined as $(f^n)(x) = (f(x))^n$, for all $x \in X$.

The function $\dfrac{1}{f}$ is called **reciprocal** of $f$ and is defined as

$\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{f(x)}$ with domain $X_2 = \{x : x \in X, f(x) \neq 0\}$.

If $c$ is any real number, then the function $cf$, called **scalar multiple** of $f$ by $c$ is defined $(cf)(x) = cf(x)$, for all $x \in X$.

**REMARK**

The functions $f$ and $g$ need not necessarily have the same domain. When their domains are different, then the operations $f + g$, $f - g$, $fg$ and $\dfrac{f}{g}$ can be performed only on a domain which is common to both the functions. Take special care of $\dfrac{f}{g}$, this operation can be performed on a domain where $g(x) \neq 0$.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Let $f(x) = x + 1$ and $g(x) = 2x - 3$ be two real functions. Find the following functions :*

(i) $f + g$ (ii) $f - g$ (iii) $fg$ (iv) $\frac{f}{g}$ (v) $f^2 - 3g$.

**Solution.** Given $f(x) = x + 1$ and $g(x) = 2x - 3$.

We note that $D_f = \mathbb{R}$ and $D_g = \mathbb{R}$, so these functions have the same domain $\mathbb{R}$.

(i) $(f + g)(x) = f(x) + g(x) = (x + 1) + (2x - 3) = 3x - 2$, for all $x \in \mathbb{R}$

(ii) $(f - g)(x) = f(x) - g(x) = (x + 1) - (2x - 3) = -x + 4$, for all $x \in \mathbb{R}$

(iii) $(fg)(x) = f(x)\, g(x) = (x + 1)(2x - 3) = 2x^2 - x - 3$, for all $x \in \mathbb{R}$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{2x-3},\ x \neq \frac{3}{2},\ x \in \mathbb{R}$

(v) $(f^2 - 3g)(x) = (f^2)(x) - (3g)(x) = (f(x))^2 - 3g(x)$

$= (x + 1)^2 - 3(2x - 3) = x^2 + 2x + 1 - 6x + 9$

$= x^2 - 4x + 10$, for all $x \in \mathbb{R}$.

**Example 2.** *If f is the identity function and g is the modulus function, then find the following functions :*

(i) $f + g$ (ii) $f - g$ (iii) $fg$ (iv) $\frac{f}{g}$.

**Solution.** As $f$ is the identity function and $g$ is the modulus function, we have $f(x) = x$, for all $x \in \mathbb{R}$ and $g(x) = |x|$, for all $x \in \mathbb{R}$.

We note that the functions $f$ and $g$ have the same domain $\mathbb{R}$.

(i) $(f + g)(x) = f(x) + g(x) = x + |x| = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0. \end{cases}$

(ii) $(f - g)(x) = f(x) - g(x) = x - |x| = \begin{cases} 0, & x \geq 0 \\ 2x, & x < 0. \end{cases}$

(iii) $(fg)(x) = f(x)\, g(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0. \end{cases}$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x}{|x|} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$

Note that $\frac{f}{g}$ is not defined at $x = 0$.

**Example 3.** *Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two real functions. Find the following functions :*

(i) $f + g$ (ii) $f - g$ (iii) $fg$ (iv) $\frac{f}{g}$.

**Solution.** Given $f(x) = \sqrt{x}$ and $g(x) = x$.

We note that $D_f = [0, \infty)$ and $D_g = \mathbb{R}$, so these functions have different domains.

Let $X = D_f \cap D_g = [0, \infty) \cap \mathbb{R} = [0, \infty)$.

Thus, the common part of the domain $= \{x : x \in \mathbb{R},\ x \geq 0\}$.

(i) $(f + g)(x) = f(x) + g(x) = \sqrt{x} + x,\ x \geq 0$

(ii) $(f - g)(x) = f(x) - g(x) = \sqrt{x} - x,\ x \geq 0$

(iii) $(fg)(x) = f(x)\,g(x) = \sqrt{x} \times x = x^{3/2}, x \geq 0$

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{x} = x^{-1/2}, x > 0.$

**Example 4.** *Let f, g be two functions defined by* $f(x) = \sqrt{x+1}$ *and* $g(x) = \sqrt{9-x^2}$, *describe the following functions :*

(i) $f + g$ (ii) $g - f$ (iii) $gf$ (iv) $\frac{f}{g}$

(v) $\frac{g}{f}$ (vi) $2f - \sqrt{5}g$ (vii) $f^2 + 7f$ (viii) $\frac{3}{g}$.

**Solution.** Given $f(x) = \sqrt{x+1}$, $g(x) = \sqrt{9-x^2}$

$\Rightarrow$ $D_f = [-1, \infty)$ and $D_g = [-3, 3]$.

Let $D = D_f \cap D_g = [-1, \infty) \cap [-3, 3] = [-1, 3] \neq \phi$, then

(i) $(f+g)(x) = f(x) + g(x) = \sqrt{x+1} + \sqrt{9-x^2}$ with domain $[-1, 3]$.

(ii) $(g-f)(x) = g(x) - f(x) = \sqrt{9-x^2} - \sqrt{x+1}$ with domain $[-1, 3]$.

(iii) $(gf)(x) = g(x)\,f(x) = \sqrt{9-x^2}\,\sqrt{x+1} = \sqrt{(9-x^2)(x+1)}$ with domain $= [-1, 3]$.

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\sqrt{9-x^2}} = \sqrt{\frac{x+1}{9-x^2}}$ with domain $D_1$

where $D_1 = D$ except where $g(x) = 0 \Rightarrow D_1 = [-1, 3)$.

(v) $\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{9-x^2}}{\sqrt{x+1}} = \sqrt{\frac{9-x^2}{x+1}}$ with domain $D_2$

where $D_2 = D$ except where $f(x) = 0 \Rightarrow D_2 = (-1, 3]$.

(vi) $(2f - \sqrt{5}g)(x) = 2f(x) - \sqrt{5}g(x) = 2\sqrt{x+1} - \sqrt{5}\sqrt{9-x^2}$ with domain $[-1, 3]$.

(vii) $(f^2 + 7f)(x) = (f(x))^2 + 7f(x) = x + 1 + 7\sqrt{x+1}$ with domain $[-1, \infty)$.

(viii) $\left(\frac{3}{g}\right)(x) = \frac{3}{g(x)} = \frac{3}{\sqrt{9-x^2}}$ with domain $D_3$

where $D_3 = D_g$ except where $g(x) = 0 \Rightarrow D_3 = (-3, 3)$.

**Example 5.** *If* $f(x) = e^x$ *and* $g(x) = \log x$, *then find the following functions :*

(i) $f + g$ (ii) $f - g$ (iii) $fg$ (iv) $\frac{f}{g}$

(v) $\frac{1}{f}$ (vi) $f^2$.

**Solution.** Given $f(x) = e^x$ and $g(x) = \log x$,

$D_f = \mathbf{R}$ and $D_g = \mathbf{R}^+$, where $\mathbf{R}^+$ is the set of positive reals.

Let $D = D_f \cap D_g = \mathbf{R} \cap \mathbf{R}^+ = \mathbf{R}^+ \neq \phi$, then

(i) $(f+g)(x) = f(x) + g(x) = e^x + \log x$, with domain $\mathbf{R}^+$.

(ii) $(f-g)(x) = f(x) - g(x) = e^x - \log x$, with domain $\mathbf{R}^+$.

(iii) $(fg)(x) = f(x)\,g(x) = e^x \log x$, with domain $\mathbf{R}^+$.

(iv) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{e^x}{\log x}$ with domain $D_1$

where $D_1 = \mathbf{R}^+$ except $g(x) = 0$ *i.e.* except $\log x = 0$ *i.e.* except ...

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(*v*) $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{e^x}$, with domain R except $f(x) = 0 = $ R.

(*vi*) $(f^2)(x) = (f(x))^2 = (e^x)^2 = e^{2x}$, with domain $D_f = $ R.

**Example 6.** *Find the domain of the function f defined by* $f(x) = \sqrt{4 - x} + \frac{1}{\sqrt{x^2 - 1}}$.

**Solution.** Let $f = g + h$, then $g(x) = \sqrt{4 - x}$ and $h(x) = \frac{1}{\sqrt{x^2 - 1}}$.

For $D_g$, $g(x)$ must be a real number $\Rightarrow 4 - x \geq 0$

$\Rightarrow \quad 4 \geq x \Rightarrow x \leq 4 \Rightarrow D_g = (-\infty, 4]$.

For $D_h$, $h(x)$ must be a real number $\Rightarrow x^2 - 1 > 0$

$\Rightarrow \quad (x + 1)(x - 1) > 0 \Rightarrow x < -1$ or $x > 1$

$\Rightarrow \quad D_h = (-\infty, -1) \cup (1, \infty)$.

As $\quad f = g + h$, so $D_f = D_g \cap D_h$

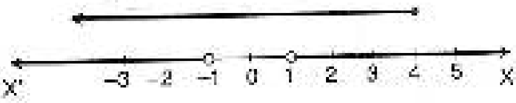



*Fig. 2.36.*

From figure, it is clear that $D_g \cap D_h = (-\infty, -1) \cup (1, 4]$.

Hence, the domain of the function $f = (-\infty, -1) \cup (1, 4]$.

**Example 7.** *If* $f(x) = \frac{x - 1}{x + 1}$, *then show that*

(*i*) $f\left(\frac{1}{x}\right) = -f(x)$ $\qquad$ (*ii*) $f\left(-\frac{1}{x}\right) = -\frac{1}{f(x)}$.

**Solution.** Given $f(x) = \frac{x - 1}{x + 1}$. Note that $D_f = $ R $- \{-1\}$.

(*i*) $f\left(\frac{1}{x}\right) = \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1} = \frac{1 - x}{1 + x} = -\frac{x - 1}{x + 1} = -f(x), \ x \neq 0, -1.$

(*ii*) $f\left(-\frac{1}{x}\right) = \frac{-\frac{1}{x} - 1}{-\frac{1}{x} + 1} = -\frac{1 + x}{-1 + x} = -\frac{x + 1}{x - 1} = -\frac{1}{f(x)}, \ x \neq 0, -1, 1.$

**Example 8.** *If* $f(x) = x - \frac{1}{x}$, *prove that* $(f(x))^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.

**Solution.** Given $f(x) = x - \frac{1}{x}$, $D_f = $ R $- \{0\}$ $\qquad$ ...(*i*)

$\Rightarrow \quad f(x^3) = x^3 - \frac{1}{x^3}$ and $f\left(\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{\frac{1}{x}} = \frac{1}{x} - x$

$\therefore \quad (f(x))^3 = \left(x - \frac{1}{x}\right)^3 = x^3 - \frac{1}{x^3} - 3x \cdot \frac{1}{x}\left(x - \frac{1}{x}\right) = x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right)$ $\qquad$ (using (*i*))

$= x^3 - \frac{1}{x^3} + 3\left(\frac{1}{x} - x\right) = f(x^3) + 3f\left(\frac{1}{x}\right)$

**Example 9.** If $f(x) = y = \frac{ax - b}{cx - a}$, then prove that $f(y) = x$.

**Solution.** Given $f(x) = y = \frac{ax - b}{cx - a}$. Note that $x \neq \frac{a}{c}$.

$$\Rightarrow \quad f(y) = \frac{ay - b}{cy - a} = \frac{a.\frac{ax - b}{cx - a} - b}{c.\frac{ax - b}{cx - a} - a}$$

$$= \frac{a(ax - b) - b(cx - a)}{c(ax - b) - a(cx - a)} = \frac{a^2x - ab - bcx + ab}{cax - bc - acx + a^2}$$

$$= \frac{a^2x - bcx}{a^2 - bc} = \frac{(a^2 - bc)x}{a^2 - bc} = x.$$

## EXERCISE 2.7

*Very short answer type questions (1 and 3) :*

1. If $f$ and $g$ are real functions defined by $f(x) = x^2 - 7$ and $g(x) = 3x + 5$, then find the values of

   (i) $f(3) - g(-5)$ (ii) $f(-2) + g(-1)$ (iii) $f\left(\frac{1}{2}\right) \times g(14)$

   (iv) $f(t) - f(-2)$ (v) $\frac{f(t) - f(5)}{t - 5}$, $t \neq 5$

2. If $f(x) = e^x$ and $g(x) = \log x$, then find :

   (i) $(f - g)(1)$ (ii) $(fg)(1)$ (iii) $\left(\frac{f}{g}\right)(3)$

3. If $f$ and $g$ are two real valued functions defined by $f(x) = 2x + 1$ and $g(x) = x^2 + 1$, then find the following functions :

   (i) $f + g$ (ii) $f - g$ (iii) $fg$ (iv) $\frac{f}{g}$

4. If $f(x) = x^2 + 1$ and $g(x) = x + 1$ be two real functions, then find the following functions :

   (i) $f + g$ (ii) $g - f$ (iii) $fg$ (iv) $\frac{f}{g}$ (v) $2g^2 - 3f$.

5. If $f(x) = \sqrt{x - 1}$ and $g(x) = 3 - 2x$ be two real functions, then find the following functions :

   (i) $f + g$ (ii) $g + f$ (iii) $fg$ (iv) $gf$ (v) $f - g$

   (vi) $\frac{f}{g}$ (vii) $\frac{g}{f}$ (viii) $\frac{3}{f}$ (ix) $3f - 2g$ (x) $2f^2 + 3g$.

   Is $f + g = g + f$ and $fg = gf$?

6. If $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt{x^2 - 1}$ be two real valued functions, then find the following functions :

   (i) $f + g$ (ii) $g - f$ (iii) $fg$ (iv) $3f - 2g$ (v) $\frac{f}{g}$

   (vi) $2f^2 + \sqrt{3}g$ (vii) $\frac{1}{f}$ (viii) $\frac{g}{f}$.

7. If $f(x) = x^3 - \frac{1}{x^3}$, prove that $f(x) + f\left(\frac{1}{x}\right) = 0$.

8. If $f(x) = x + \frac{1}{x}$, prove that $(f(x))^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$.

9. If $y = f(x) = \frac{6x - 5}{5x - 6}$, prove that $f(y) = x$, $x \neq \frac{6}{5}$.

## CHAPTER TEST

1. If A = {1, 2, 3}, B = {4, 5} and C = {5, 6}, then verify that
   (i) A × (B ∪ C) = (A × B) ∪ (A × C) (ii) A × (B ∩ C) = (A × B) ∩ (A × C)
   (iii) A × (B – C) = (A × B) – (A × C).
2. Let A = {2, 4, 6, 8} and B = {0, 6, 8, 9, 10}. Find the elements of (A ∩ B) × (A – B) corresponding to the relation '*is a multiple of*'.
3. Let A = {6, 7, 8, 10}, B = {2, 4, 5}, $a \in$ A, $b \in$ B and R be the relation from A to B defined by $a$ R $b$ if and only if $a$ is divisible by $b$. Write R in the roster form.
4. Let R = $\{(x, y) ; x + 2y < 6, x, y \in N\}$
   (i) Find the domain and the range of R (ii) Write R as a set of ordered pairs.
5. Let R = $\{(x, y) ; y = x + 1$ and $y \in \{0, 1, 2, 3, 4, 5\}\}$.
   (i) List the elements of R. (ii) Represent R by an arrow diagram.
6. Let $f$ be the subset of Q × Z defined by $f = \left\{\left(\frac{m}{n}, m\right) : m, n \in Z, n \neq 0\right\}$. Is $f$ a function from Q to Z? Justify your answer.
   **Hint.** $f\left(\frac{1}{2}\right) = 1$ and $f\left(\frac{2}{4}\right) = 2$ but $\frac{1}{2} = \frac{2}{4}$.
7. Let $f : X \to Y$ be defined by $f(x) = x^2$ for all $x \in X$ where X = {–2, –1, 0, 1, 2, 3} and Y = {0, 1, 4, 7, 9, 10}.
   Write the relation $f$ in the roster form. Is $f$ a function?
8. Is $g$ = {(1, 1), (2, 3), (3, 5), (4, 7)} a function? If this is described by the relation $g(x) = \alpha x + \beta$, then what values should be assigned to α and β.
9. Consider the function $f(x) = x + \frac{1}{x}$, $x \in$ R, $x \neq 0$. Is $f$ one-one?
   **Hint.** Check that $f(2) = f\left(\frac{1}{2}\right)$.
10. Prove that the function $f : N \to N$ defined by $f(x) = 3x - 2$ is one-one but not onto.
    **Hint.** 2 has no pre-image in N.
11. Find the domain of the function $f$ given by $f(x) = \frac{1}{\sqrt{|x| - x}}$.
12. Determine a quadratic function '$f$' defined by
    $f(x) = ax^2 + bx + c$ if $f(0) = 6$, $f(2) = 11$ and $f(-3) = 6$.
13. Find the domain and the range of the function $f(x) = 2 - 3x^2$. Also find $f(-2)$ and the numbers which are associated with the number –25 in its range.
14. Find the domain and the range of the following functions :
    (i) $\sqrt{x - 3}$ (ii) $\sqrt{25 - x^2}$ (iii) $5 - |x + 1|$.
15. Draw the graph of the function $f(x) = \begin{cases} 1 + 2x & , x < 0 \\ 3 + 5x & , x \geq 0 \end{cases}$.
    Hence, find its range.
16. If $f(x) = 2x + 5$ and $g(x) = x^2 - 1$ are two real valued functions, find the following functions :
    (i) $f + g$ (ii) $f - g$ (iii) $fg$ (iv) $\frac{f}{g}$ (v) $\frac{g}{f}$ (vi) $3g + 2f^2$.

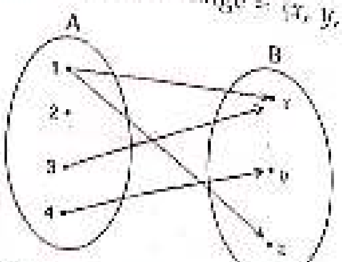
# ANSWERS

## EXERCISE 2.1

1. (*i*) $a = 2, b = 3$ (*ii*) $a = 2, b = 1$ (*iii*) $a = 3, b = -1$ (*iv*) $a = ...$
2. (*i*) $x = 8, y = 2$ (*ii*) $x = \frac{11}{3}, y = \frac{2}{3}$ 3. $a = 1, b = 7$
4. $P \times Q = \{(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)\}$ and $Q \times P = \{(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)\}$
5. (*i*) $\{(-1, 3), (-1, 5), (0, 3), (0, 5), (1, 3), (1, 5)\}$ (*ii*) $\{(3, -1), (3, 0), (3, 1), (5, -1), (5, 0), (5, 1)\}$ (*iii*) $\{(3, 3), (3, 5), (5, 3), (5, ...$
6. 6 7. 9 8. (*i*) 20 (*ii*) 20 (*iii*) 16 (*iv*) 25
9. 16 10. $\{(2, 4), (2, 6), (3, 4), (3, 6), (5, 6)\}$
11. $\{(-1, 6), (2, 3), (5, 0)\}$ 12. $\{(2, 4), (2, 6), (2, 10), (3, 6), (3, 9), (4, 4) ...$
13. $S = \{(-1, 0), (0, 0), (1, 0), (2, 0), (3, 0)\}$ 14. $A = \{p, m\}$ and $B = \{3, ...$
15. $A = \{-1, 2, 3\}$ and $B = \{1, 2\}$ 16. $B = \{1, 2\}$
17. $B \times A = \{(1, x), (2, y), (3, x), (3, y), (1, y), (2, x)\}$
18. (*i*) $\{(1, 5), (1, 7), (1, 9), (2, 5), (2, 7), (2, 9), (3, 5), (3, 7), (3, 9), (4, 5), (4, 7), (4, 9) ...$ (*ii*) $\{(5, 1), (5, 2), (5, 3), (5, 4), (7, 1), (7, 2), (7, 3), (7, 4), (9, 1), (9, 2), (9, 3), (9, 4) ...$ (*iii*) No (*iv*) Yes
19. (*i*) $\{(1, 4), (2, 4), (3, 4)\}$ (*ii*) $\{(1, 4), (2, 4), (3, 4)\}$ (*iii*) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6) ...$ (*iv*) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6) ...$
21. (*i*) $\{(0, 3), (1, 3)\}$ (*ii*) $\{(0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5)\}$
23. $\{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$
24. $A \times B = \{(-1, 2), (-1, 3), (2, 2), (2, 3), (4, 2), (4, 3)\}$, $B \times A = \{(2, -1), (3, -1), (2, 2), (3, 2), (2, 4), (3, 4)\}$.
25. $A = \{a, b, c\}$ ; $(a, 3), (a, 5), (b, 2), (b, 5), (c, 2), (c, 3)$

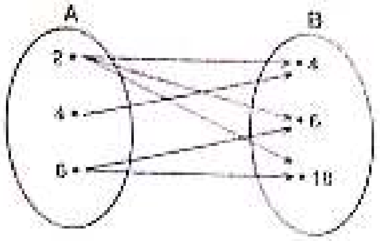
## EXERCISE 2.2

1. (*i*) 64 (*ii*) 64 (*iii*) 16 2. $\{(1, 3), (1, 4), (2, 3), (2, 4)\}$; 16
3. (*i*) 16 (*ii*) 64 (*iii*) 64 (*iv*) 512 4. Domain $= \{-2, 0, 3, 4, 5\}$ and range $= \{1, 2 ...$
5. $R = \{(2, 4), (2, 6), (3, 4), (3, 6), (5, 6)\}$ 6. $R = \{(1, 2), (3, 4), (5, 6), (7 ...$
7. $R = \{(2, 4), (2, 6), (2, 10), (3, 6), (3, 9), (4, 4)\}$ 8. $\{(2, 3), (3, 4), (4, 5), (5, 6)\}$
9. $R = \{(1, 3), (2, 6), (3, 9), (4, 12), (5, 15)\}$
10. (*i*) $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$ (*ii*) $R = \{(0, 4), (1, 9), (3, 25), (5, ...$
11. (*i*) Domain $= \{1, 3, 4\}$ and range $= \{x, y, z\}$ (*ii*)




12. (*i*) Domain of $R = \{1, 2, 3, 4, 5\}$, range of $R = \{-1, 0, 1, 2, 3\}$ (*ii*) $R = \{(x, y) : x \in N, 1 \le x \le 5, y = x - 2\}$

**13.** R = {(2, 4), (2, 6), (2, 18), (4, 4), (6, 6), (6, 18)}




**14.** (*i*) and (*iii*)

**15.** Domain = {0, 1, 2, 3, 4, 5} and range = {5, 6, 7, 8, 9, 10}

**16.** (*i*) {(5, 3), (6, 4), (7, 5)}

(*ii*) $\{(x, y) : y = x - 2,\ x \in N,\ 5 \le x \le 7\}$

Domain = {5, 6, 7} and range = {3, 4, 5}

**17.** (*i*) Domain = {1, 2, 3, ..., 9} and range = {9, 8, 7, ..., 1}

(*ii*) Domain = {1, 2, 3, 4} and range = {3}

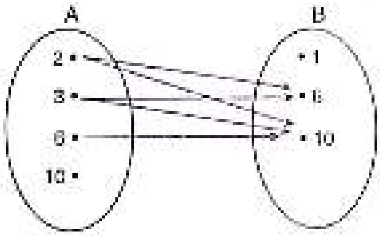
**18.** (*i*) {(9, 1), (6, 2), (3, 3)} (*ii*) {9, 6, 3} (*iii*) {1, 2, 3}

**19.** Domain of R = {0, 6, 8, 10} and range of R = {0, 6, 8, 10}; R = {(0, 10), (6, 8), (8, 6), (10, 0)}

**20.** (*i*) R = {(1, 13), (2, 8), (3, 7), (4, 7), (6, 8), (12, 13)}

(*ii*) {1, 2, 3, 4, 6, 12} (*iii*) {13, 8, 7}

**21.** (2, 6), (2, 10), (3, 6), (3, 10), (6, 10)




**22.** Domain = {3, 6, 9} and range = {1, 2, 3}

**23.** (*i*) (2, 1), (2, 0), (2, −1), (2, −2), (1, 0), (1, −1), (1, −2), (0, −1), (0, −2), (−1, −2)

(*ii*) (0, 0), (1, −1), (1, 1) (*iii*) (−2, 2), (−1, 1), (2, −2), (1, −1), (0, 0)

**24.** (*i*) {(1, 3), (1, 5), (3, 3), (3, 5), (5, 3), (5, 5), (6, 4)}

(*ii*) {(1, 3), (1, 5), (3, 3), (3, 5), (5, 3), (5, 5)}

**25.** $a = 1,\ b = \pm 2$ **26.** (*i*) $y = 3x + 2$ (*ii*) $y = 2 - 3x$

## EXERCISE 2.3

**1.** (*i*) No (*ii*) Yes; domain = {*a*, *b*, *c*, *d*} and range = {*b*, *c*, *d*, *e*}

(*iii*) No (*iv*) Yes; domain = {1, 2, 3, 4} and range = {2}

(*v*) No (*vi*) Yes; domain = {2, 4, 6, 8, 10, 14} and range = {1, 2, 3, 4, 7}

**2.** (*ii*). **3.** (*i*) Yes (*ii*) No **4.** (*i*) Yes (*ii*) Yes (*iii*) Yes

**5.** (*i*) {(−1, 2), (0, 1), (1, 2), (3, 10)} (*ii*) {−5, −1, 5, 9}

**6.** (*i*) and (*iv*) **7.** Yes; range = {0, 2, 7}

**8.** {(1, 4), (2, 1), (3, 4), (4, 3), (5, 4)}; yes, because each element of the domain has a unique image in the codomain

**9.** {(−1, 1), (0, 0), (1, 1), (2, 4), (−2, 4)}; yes; $f(x) = x^2,\ x \in A$

**10.** {(*p*, 3), (*q*, 7), (*r*, 7), (*s*, 1)} : domain = {*p*, *q*, *r*, *s*}, codomain = {1, 3, 5, 7}, range = {1, 3, 7}; yes

**11.** (*i*) {−2, −1, 1, 2} (*ii*) $\left\{-\frac{1}{2}, -1, 1, \frac{1}{2}\right\}$ (*iii*) Yes

12. $f = \{(-1, 0), (0, 1), (3, 28), (9, 730), (7, 344)\}$; range of $f = \{0, 1, 28, 730, 344\}$

13. $a = 2, b = -3$ 14. $a = \frac{4}{3}, b = 10$ 15. $a = 3, b = -2$

21. (*i*) Bijective (*ii*) bijective

## EXERCISE 2.4

1.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $y = f(x)$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 |

2. 21 3. $-9$

4. No; because $f(3) = 3^2 = 9$, also $f(3) = 2 \times 3 = 6$, so the element 3 of the domain of $f$ has two images

5. $-49$ 6. $3, -3$ 7. 6 8. 80; 3, $-3$ 9. (*i*) 5 (*ii*) 4, $-4$

10. (*i*) $\{1, 3, 9, 19\}$ (*ii*) 171 (*iii*) $-2$

11. (*i*) $\{4, 7, 12, 19, 28\}$ (*ii*) 196 (*iii*) No (*iv*) 2

12. (*i*) $\mathbb{R} - \{2\}$ (*ii*) $\mathbb{R} - \{28\}$ (*iii*) $\mathbb{R} - \{-1, -2\}$ (*iv*) $\mathbb{R} - \{-1, 1\}$

13. (*i*) $[2, \infty)$ (*ii*) $\left(-\infty, \frac{2}{3}\right]$ (*iii*) $(1, \infty)$ (*iv*) $(-\infty, 3)$

14. (*i*) $\mathbb{R}$ (*ii*) $\mathbb{R}$ (*iii*) $\mathbb{R}$ (*iv*) $\mathbb{R} - \{0\}$

15. (*i*) $[2, \infty)$ (*ii*) $(-\infty, 1]$ 16. $D_f = \mathbb{R}$, $R_f = [1, \infty)$; 9; 5, $-5$

17. (*i*) $[-2, \infty)$; $[0, \infty)$ (*ii*) $\left(-\infty, \frac{3}{2}\right]$; $[0, \infty)$ (*iii*) $(5, \infty)$; $(0, \infty)$

18. (*i*) $\mathbb{R} - \{4\}$; $\{-1\}$ (*ii*) $\mathbb{R} - \{3\}$; $\mathbb{R} - \{6\}$ (*iii*) $\mathbb{R} - \left\{-\frac{1}{2}\right\}$; $\mathbb{R} - \left\{\frac{1}{2}\right\}$
(*iv*) $\mathbb{R} - \{4\}$; $\mathbb{R} - \{-1\}$

19. (*i*) $[-4, 4]$; $[0, 4]$ (*ii*) $(-\infty, -3] \cup [3, \infty)$; $[0, \infty)$ (*iii*) $(-2, 2)$; $\left[\frac{1}{2}, \infty\right)$

20. (*i*) $\mathbb{R}$; $[0, \infty)$ (*ii*) $\mathbb{R}$; $(-\infty, 3]$ 21. $\mathbb{R} - \{0\}$; $\{0, -1\}$

22. Domain of $f = \mathbb{R} - \{4\}$; range of $f = \{1, -1\}$

23. (*i*) $x = 4$ (*ii*) $x > 4$ 24. $\left\{-1, \frac{4}{3}\right\}$

## EXERCISE 2.5

1. (*i*) $\mathbb{R}$ (*ii*) $\mathbb{R}$.

2. (*i*) $\mathbb{R} - \{4\}$ (*ii*) $\mathbb{R}$

3. (*i*) $(-\infty, 1) \cup [7, \infty)$ (*ii*) $[-1, 1]$

4. (*i*) $[0, \infty)$ (*ii*) $(-\infty, 1]$

## EXERCISE 2.6

1. (*i*) $\left[4, \frac{9}{2}\right]$ (*ii*) $\left[-\frac{3}{2}, -1\right)$ (*iii*) $\left(-\frac{4}{3}, -1\right]$

3. (*i*) $\left(\frac{3}{4}, \infty\right)$ (*ii*) $(-3, 3) - \{-2\sqrt{2}, 2\sqrt{2}\}$

## EXERCISE 2.7

1. (*i*) 6 (*ii*) 13 (*iii*) $\frac{1363}{4}$ (*iv*) $t^2 - 4$ (*v*) $t + 5$

2. (*i*) $e$ (*ii*) 0 (*iii*) $\frac{e^3}{\log 3}$

3. (*i*) $x^2 + 2x + 2, x \in \mathbf{R}$ (*ii*) $2x - x^2, x \in \mathbf{R}$

(*iii*) $2x^3 + x^2 + 2x + 1, x \in \mathbf{R}$ (*iv*) $\frac{2x+1}{x^2+1}, x \in \mathbf{R}$

4. (*i*) $x^3 + x + 2, x \in \mathbf{R}$ (*ii*) $x - x^3, x \in \mathbf{R}$ (*iii*) $x^4 + x^3 + x + 1, x \in \mathbf{R}$

(*iv*) $\frac{x^3+1}{x+1}, x \in \mathbf{R}, x \neq -1$ (*v*) $2x^2 + 4x - 3x^3 - 1, x \in \mathbf{R}$

5. (*i*) $\sqrt{x-1} + 3 - 2x, x \geq 1$ (*ii*) $3 - 2x + \sqrt{x-1}, x \geq 1$ (*iii*) $\sqrt{x-1}\,(3 - 2x), x \geq 1$

(*iv*) $(3 - 2x)\sqrt{x-1}, x \geq 1$ (*v*) $\sqrt{x-1} - 3 + 2x, x \geq 1$ (*vi*) $\frac{\sqrt{x-1}}{3-2x}, x \geq 1, x \neq \frac{3}{2}$

(*vii*) $\frac{3-2x}{\sqrt{x-1}}, x > 1$ (*viii*) $\frac{3}{\sqrt{x-1}}, x > 1$ (*ix*) $3\sqrt{x-1} - 6 + 4x, x \geq 1$

(*x*) $-4x + 7, x \geq 1$; Yes and yes

6. (*i*) $\sqrt{x-2} + \sqrt{x^2-1}$ (*ii*) $\sqrt{x^2-1} - \sqrt{x-2}$ (*iii*) $\sqrt{(x-2)(x^2-1)}$

(*iv*) $3\sqrt{x-2} - 2\sqrt{x^2-1}$ (*v*) $\sqrt{\frac{x-2}{x^2-1}}$ (*vi*) $2(x-2) + \sqrt{3(x^2-1)}$

(*vii*) $\frac{1}{\sqrt{x-2}}$ (*viii*) $\sqrt{\frac{x^2-1}{x-2}}$

Domain in case (*i*) to (*vi*) is $[2, \infty)$ and in cases (*vii*), (*viii*) is $(2, \infty)$

## CHAPTER TEST

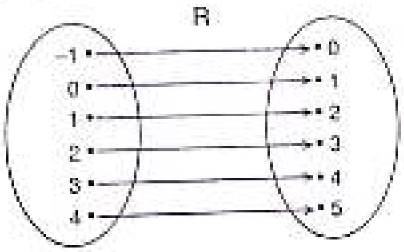
2. (6, 2), (8, 2), (8, 4)

3. {(6, 2), (8, 2), (8, 4), (10, 2), (10, 5)}

4. (*i*) Domain of R = {1, 2, 3} and range of R = {1, 2} (*ii*) {(1, 1), (1, 2), (2, 1), (3, 1)}

5. (*i*) {(−1, 0), (0, 1), (1, 2), (2, 3), (3, 4), (4, 5)}

(*ii*)




6. No

7. {(−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4), (3, 9)}; yes

8. Yes; $\alpha = 2, \beta = -1$

9. No

11. $(-\infty, 0)$

12. $f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + 6$

13. R; $(-\infty, 2]$; −10; 3, −3

14. (*i*) $[3, \infty)$; $[0, \infty)$ (*ii*) $[-5, 5]$; $[0, 5]$ (*iii*) R; $(-\infty, 5]$

15. $(-\infty, 1) \cup [3, \infty)$

16. (*i*) $x^2 + 2x + 4, x \in \mathbf{R}$ (*ii*) $2x + 6 - x^2, x \in \mathbf{R}$

(*iii*) $2x^3 + 5x^2 - 2x - 5, x \in \mathbf{R}$ (*iv*) $\frac{2x+5}{x^2-1}, x \in \mathbf{R}, x \neq 1, -1$

(*v*) $\frac{x^2-1}{2x+5}, x \in \mathbf{R}, x \neq -\frac{5}{2}$ (*vi*) $11x^2 + 40x + 47, x \in \mathbf{R}$